Dividing Squares into Triangles of Equal Area

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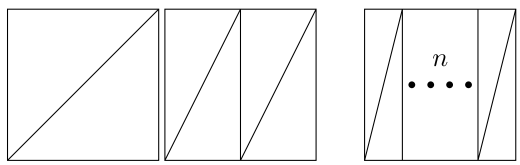
1 Introduction

Consider a square. It is easy to divide a square into 2 triangles of equal area. Similarly, it is easy to divide a square into 4 triangeles of equal area. With a little bit of thought, it can be seen that we may divide a square into $2n$ triangles of equal area for any n. This may be easy, but is it possible to divide a triangle into



(a) 2 (b) 4 (c) $2n$

Figure 1: Dividing squares into triangles of equal area

an odd number of triangles of equal area? In the course of this talk we will use techniques from the theory of valuations and combinatorial geometry to show that this is impossible.

2 Valuations and Extensions

We will introduce a concept from number theory that may seem completely unrelated to the problem at hand.

Definition 1. Given a ring R, we define a *valuation* to be a function $v : R \to \mathbb{R} \cup \{-\infty\}$ that satisfies the following properties:

1. If $x \in R$ then $v(x) = -\infty$ if and only if $x = 0$
2. For all $x, y \in R$, $v(xy) = v(x) + v(y)$
3. For all $x, y \in R$, $v(x + y) \geq \min(v(x), v(y))$

There are a number of examples of these valuations, but we will only consider the following, called the *2-adic valuation.* Let $\frac{p}{q} \in \mathbb{Q}$ such that

$$\frac{p}{q} = 2^r \frac{p'}{q'} \quad \text{and} \quad 2 \nmid pq$$

Then we define $v_2(\frac{p}{q}) = r$ and let $v_2(0) = -\infty$. We leave it as an (easy) exercise to check that this is, indeed, a valuation as defined above. There are a few important properties of valuations that are required.

Proposition 2. *If v is a valuation on R then the following hold:*

1. $v(1) = 0$

2. $v(x^{-1}) = -v(x)$

3. $v(-x) = v(x)$

4. If $v(x) < v(y)$ *then* $v(x+y) = v(x)$

Proof. We prove each statement individualy:

1. This is a direct computation. $v(x) = v(1 \cdot x) = v(x) + v(1)$ and the result follows.

2. A similar computation applies: $0 = v(1) = v(x \cdot x^{-1}) = v(x) + v(x^{-1})$.

3. Note that R has characteristic 0. By above it suffices to show that $v(-1) = 0$ But $0 = v(1) = v((-1)^2) = 2v(-1)$.

4. Note that $v(x+y) \geq v(x)$ and that $v(x) = v(x+y-y) \geq \min(v(x+y), v(y)) = v(x+y)$.

■

Note that we have defined our valuation over $\mathbb{Q}$ but we will eventually wish to have a valuation over $\mathbb{R}$. For some numbers, it is obvious. For instance, we have $1 = v_2(2) = v_2(\sqrt{2}^2) = 2 \cdot v_2(2)$ so $v_2(\sqrt{2}) = \frac{1}{2}$. A similar argument applies to other rational powers of 2 and so we may naturally extend our valuation to $\mathbb{Q}[2^{\mathbb{Q}}]$. Note that this is not $\mathbb{R}$. How might we define $v_2(\log 3)$ or $v_2(\pi)$ or $v_2(e)$? To do this we will need a theorem on extensions. We need another definition for this. Note that this requires some more advanced machinery and is included only for the sake of completeness. We have

Definition 3. Let $R \subset K$ be a subring of a field K. We say that R is a *valuation ring* if and only if any (and hence all) of the following equivalent statements are satisfied:

1. $0 \in R$ and if $x \in K^{\times}$ and $x \notin R$ then $x^{-1} \in R$

2. There exists a valuation v on K such that $R = \{x \in K | v(x) \geq 0\} \cup \{0\}$

Given a prime ideal $\mathfrak{p} \subset R$, we define the $\mathfrak{p}$-adic valuation $v_{\mathfrak{p}}$ on R such that $v_{\mathfrak{p}}(x) = r$ if and only if $x \in \mathfrak{p}^r \setminus \mathfrak{p}^{r+1}$ and $v_{\mathfrak{p}}(0) = -\infty$. We leave it as an (easy) exercise to show that this is, in fact, a valuation. Moreover, valuation rings are local and their valuation is given by the $\mathfrak{p}$-adic valuation for their unique maximal ideal $\mathfrak{p}$.

Remark 4. The fact that the statements defining a valuation ring are equivalent requires proof, but this is easy and we leave it as an exercise.

We will need a result from ring theory. In order to prove this, we assume knowledge of Zorn's lemma.

Theorem 5 (Chevalley)**.** *Let $R \subset K$ be a valuation ring of a field K and let $\mathfrak{p} \subset R$ be a prime ideal. Then there is a valuation ring $S \subset K$ and prime ideal $\mathfrak{m} \subset S$ such that $R \subset S$ and $\mathfrak{m} \cap R = \mathfrak{p}$.*

Proof. Let

$$S = \{(A, \mathfrak{a}) | R_{\mathfrak{p}} \subset A \subset K, \ \mathfrak{p}R_{\mathfrak{p}} \subset \mathfrak{a} \subset A\}$$

Note first that S is nonempty because $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}}) \in S$. We may introduce a partial order on S in such a way that $(A, \mathfrak{a}) \leq (A', \mathfrak{a}')$ if and only if $A \subset A'$ and $\mathfrak{a} \subset \mathfrak{a}'$. Note that any chain $\{(A_i, \mathfrak{a}_i)\}$ such that $(A_i, \mathfrak{a}_i) \leq (A_{i+1}, \mathfrak{a}_{i+1})$ for all i is bounded above by $(\bigcup_i A_i, \bigcup_i \mathfrak{a}_i)$ and so we may apply Zorn's lemma to get $(M, \mathfrak{m})$ a maximal element of S. By maximality we see that M is local with maximal ideal $\mathfrak{m}$. By construction we see that $\mathfrak{m} \cap R_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$ and so by elementary commutative algebra we have that $\mathfrak{m} \cap R = \mathfrak{p}$. Thus it suffices to show that M is a valuation ring. Suppose $x, x^{-1} \in K^{\times}$ such that $x, x^{-1} \notin M$. By maximality, we must have $\mathfrak{m}[x] = M[x]$ and $\mathfrak{m}[x^{-1}] = M[x^{-1}]$. Then in particular we have $1 \in \mathfrak{m}[x]$ and $1 \in \mathfrak{m}[x^{-1}]$ and so there exist minimal m, n and elements $\alpha_0, \alpha_1, \ldots, \alpha_m, \beta_0, \beta_1, \ldots, \beta_n \in \mathfrak{m}$ such that

$$1 = \sum_{i=0}^{m} \alpha_i x^i = \sum_{j=0}^{n} \beta_j x^{-j}$$

Suppose that we have $m \geq n$. We have that $\beta_0 \in \mathfrak{m}$ and we have that M is local so $1-\beta_0 \in M^\times$ by standard commutative algebra. Let $\gamma_j = \frac{\beta_j}{1-\beta_0}$. Then we have

$$1 = \sum_{j=1}^{n} \gamma_j x^{-j} \qquad \text{and so} \qquad x^m = \sum_{j=1}^{n} \gamma_j x^{m-j}$$

Then we have

$$1 = \sum_{i=0}^{m} \alpha_i x^i = \sum_{i=0}^{m-1} \alpha_i x^i + \alpha_m \sum_{j=1}^{n} \gamma_j x^{m-j} = \sum_{i=0}^{m-1} \delta_i x^i$$

because $m \geq n$. This contradicts the minimality of m. If $m < n$ we may switch x and x^{-1} and apply the same argument. ■

Given Theorem 5, we may then prove the existence of an extension of a valuation.

Proposition 6. *Let K_2/K_1 be an algebraic extension and let $R_1 \subset K_1$ be a valuation ring with maximal ideal $\mathfrak{p}_1$. Then there exists a valuation ring $R_2 \subset K_2$ with maximal ideal $\mathfrak{p}_2$ such that R_2 is an algebraic extension of R_1 and the valuation on R_2, v_2, extends the valuation on R_1, v_1, such that $v_2|_{R_1} = v_1$.*

Proof. This is immediate from Theorem 5 and the fact that a valuation ring uniquely defines a valuation. ■

Now note that we may compose these extensions and take the categorical colimit to get an extension of v_2 to $\mathbb{C}$; restricting to $\mathbb{R}$ yields the following:

Proposition 7. *There exists an extension $\widetilde{v}_2 : \mathbb{R} \to \mathbb{R}$ of v_2 such that $\widetilde{v}_2|_{\mathbb{Q}} = v_2$. In the sequel we will simply write this as v_2.*

Proof. We may p-adically complete $\mathbb{Q}$ to $\mathbb{Q}_2$. Now we have that $\overline{\mathbb{Q}}_2/\mathbb{Q}_2$ is an (infinite) algebraic extension. Thus Theorem 5 applies and there is an extension of our valuation to $\overline{\mathbb{Q}}_2$. Recall the faact that there is an abstract isomorphism of fields $\mathbb{C} \cong \overline{\mathbb{Q}}_2$. This induces a valuation v_2 on $\mathbb{C}$. We may then restrict $v_2|_{\mathbb{R}}$ and we win. ■

Finally we note that we may define the *2-adic metric* as $|\cdot|_2$ such that $|x|_2 = 2^{-v_2(x)}$. It is an elementary exercise to check that this does indeed induce a metric. We summarize some properties in the following proposition:

Proposition 8. *The following is true about the 2-adic metric $|\cdot|_2$:*

1. $|1|_2 = 1$ *and* $|0|_2 = 0$
2. *For all $x \in \mathbb{R}^\times$, $|x^{-1}|_2 = |x|_2^{-1}$ and moreover, $|xy|_2 = |x|_2|y|_2$*
3. *For all $x \in \mathbb{R}$, $|-x|_2 = |x|_2$*
4. *If $|x|_2 < |y|_2$ then $|x+y|_2 = |y|_2$*

Proof. These all follow from Proposition 2 and elementary properties of exponentiation. ■

With the theory of valuations in hand, we digress to something completely different: combinatorial geometry.

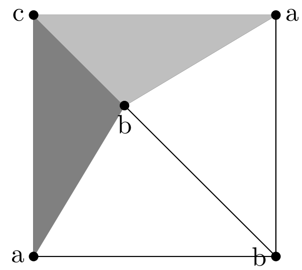


Figure 2: A colored triangulated square. The shaded triangles are full.

3 Coloring, Triangulations, and Sperner's Lemma

We first introduce the concept of triangulation.

Definition 9. Given a polygon P in the plane, a *triangulation* is a set of vertices in the polygon connected to partition P into triangles. A coloring of the triangles, is a function $c : V \to \{a, b, c\}$ where V is the set of vertices. We say that a triangle is *full* if each of its vertices is colored a different color. We say that an edge is an ab-edge if its endpoints are colored a, b respectively with other colored edges defined similarly.

Remark 10. Note that a coloring can simply be thought of as assigning a 'color' to each vertex.

The following lemma is the key to our proof.

Lemma 11 (Sperner)**.** *Let T be a triangulation of some polygon P and let F be the number of full triangles. Let A be the number of ab-edges on the boundary of P. Then $A \equiv F \pmod 2$.*

Proof. For our triangulation, for each ab-edge put a dot on each side of the edge and let N be the number of dots in the interior of the polygon P. We will show that $N \equiv F$ and $N \equiv A \pmod 2$, which will suffice by transitivity. Note that given a triangle Δ in the triangulation, if the triangle is not full then there are either 0 or 2 dots in its interior. If the triangle is full then there is only one dot in its interior. But N is the sum of the number of dots in all of the triangles so $N \equiv F \pmod 2$. For each edge on the boundary of P, if the edge is an ab-edge then it contributes one dot and otherwise it contributes zero dots. Thus $F \equiv N \pmod 2$ and we win. ∎

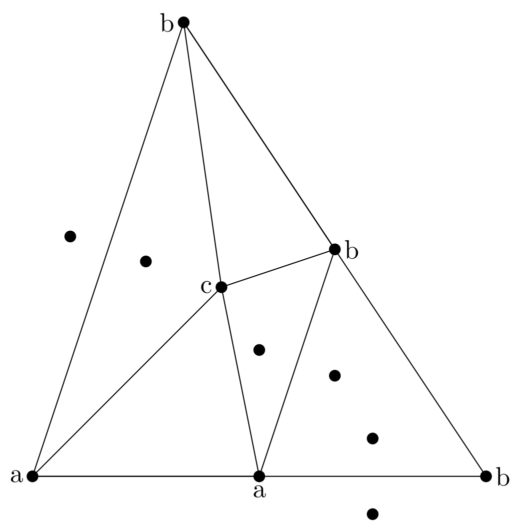


Figure 3: A colored triangle with the dots as in Lemma 11. Note that there are 4 interior dots, 2 perimiter ab-edges, and 2 full triangles

4 Monsky's Theorem

We are now able to prove our original conjecture. In order to do this, we color the entire plane $\mathbb{R}^2$ as follows:

$$S_a = \{(x, y) : |x|_2, |y|_2 < 1\}$$
$$S_b = \{(x, y) : |x|_2 \geq 1, \ \ |x|_2 \geq |y|_2\}$$
$$S_c = \{(x, y) : |y_2| \geq 1, |y|_2 > |x|_2\}$$

It is an easy exercise to see that we have colored every point in the plane, using Proposition 7 to extend our absolute value from $\mathbb{Q}$ to $\mathbb{R}$. We need a lemma.

Lemma 12. *Let Δ be a full triangle in the plane and let A be its area. Then $|A|_2 > 1$*

Proof. Note that by Proposition 8.4, we may translate Δ to set the a colored vertex to the origin. Thus we may assume without loss of generality that the vertices of Δ are $(0, 0)$, (x_b, y_b), and (x_c, y_c). As we know,

$$A = \pm \frac{x_b y_c - x_c y_b}{2}$$

and, by Proposition 8.3, we may drop the sign. By both parts of Proposition 8.2, we have $|A|_2 = 2|x_b y_c - x_c y_b|_2$ because $|\frac{1}{2}|_2 = |2|_2^{-1} = 2$. Now, we have by the coloring that $|x_b|_2 \geq |y_b|_2$ and $|y_c|_2 > |x_c|_2$. Thus, by multiplicativity we have $|x_b y_c|_2 > |x_c y_b|_2$ and so again by Proposition 8.4, we have $|A|_2 = 2|x_b y_c|_2$. But we have that $|x_b|_2, |y_c|_2 \geq 1$ by definition of the coloring so their product is at least one and $|A|_2 \geq 2$ as desired. ■

We are finally ready to prove the result.

Theorem 13 (Monsky)**.** *A square can be divided into m triangles of equal area if and only if m is even.*

Proof. Note that in the case of m even, Figure 1 provides the general technique. Thus we only have to prove the only if part of the statement. Because area is translation invariant and scaling does not change area ratios, we may assume that our square, $\Sigma = [0, 1]^2$. Let T be a triangulation of Σ into m triangles of equal area with coloring inherited from the coloring of $\mathbb{R}^2$. Note that the coloring of the vertices of the square is as in Figure 4. Now note that any point on $[0, 1] \times \{0\}$ cannot be color c. We may induct on the number

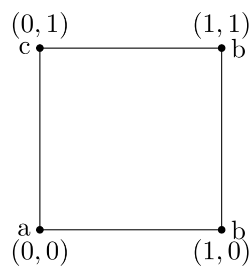


Figure 4: The colors of the four vertices of the unit square

of vertices on this segment to see that there is an odd number of ab-edges on this segment. Note, too, that there can be no b colored points on $\{0\} \times [0, 1]$, no a colored points on $\{1\} \times [0, 1]$ and no a colored points on $[0, 1] \times \{1\}$. Thus we have an odd number of ab-edges on the boundary of Σ. By Lemma 11, there is at least one full triangle. By Lemma 12, this triangle has area A such that $|A|_2 > 1$. But all triangles have the same area and the sum of these areas is the area of the square, so we have $m \cdot A = 1$. Taking absolute values we see that $|m|_2 \cdot |A|_2 = 1$ and so we must have $|m|_2 < 1$. But m is an integer so $|m|_2 < 1$ implies that $2|m$ and so we are done. ■

A few concluding remarks are in order. First is that Sperner's lemma may be nicely applied to other interesting results, including Brouwer's Fixed Point Theorem in dimension 2. Next is that there are many generalizations to Monsky's result. A few include the fact that dividing a hyper-cube of dimension n into m simplices of equal volume can only be accomplished if $n!|m$, with a proof method similar to the above. There are also results on when a regular n-gon can be triangulated into triangles of equal area and more complicated shapes, all using the beautiful methods of Monsky.