Intersection on a Nonsingular Variety

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These notes are for a talk given for the Reading Seminar on Intersection Theory for graduate students at Columbia in the Fall 2018 semester. These notes are entirely based on [Ful98, Chapter 8] and the reader is referred there for a clearer and more eloquent exposition.

1 A Brief Review

Recall our general setup. Let $i : X \to Y$ be a regular imbedding of codimension d and let $f : V \to Y$ a morphism such that V is a purely k dimensional scheme. This gives the fibre square

$$\begin{tikzcd} W \arrow[r, "j"] \arrow[d, "g"] & V \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

Denote $N_X Y$ be the normal bundle of X in Y and let $N = g^* N_X Y$ be the pull back to W. Then N is a rank d bundle on W with projection $\pi : N \to W$. Let $C = C_W V$ be the normal cone of W in V. Recall that we have

$$\begin{tikzcd} C \arrow[rr, hook] \arrow[dr] & & N \arrow[dl, "\pi"] \\ & W & \end{tikzcd}$$

with C purely k-dimensional. Thus $[C] \in A_k N$. Let $s : W \to N$ denote the zero section. Then we may define $X \cdot V = s^*[C] \in A_{k-d} W$. This gives the intersection product.

We also have a refined version of the above construction. Similarly, we have $i : X \to Y$ be a regular imbedding of codimension d and let $f : Y' \to Y$ be a morphism. This gives the following fiber square

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "g"] & Y' \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

We define

$$i^! : Z_k Y' \to Z_{k-d} X' \qquad\qquad [V] \mapsto X \cdot V$$

with linear extension. This passes to the Chow groups and we have a refined intersection product

$$i^! : A_k Y' \to A_{k-d} X'$$

Now, let k a field. We say that Y/k is non singular if it smooth over k.

Proposition 1. *If $f : X \to Y$ is a smooth morphism of relative dimension n and $\gamma : X \to X \times Y$ is the graph morphism, then γ is a regular imbedding of codimension n with normal bndle f^*T_Y. In particular, if X/k is smooth, then the diagonal $\delta : X \to X \times X$ is a regular imbedding of codimension equal to* $\dim X$

Proof. See [Ful98, Appendix B.7.3]. ■

We are now ready to move on to the main body of the talk.

2 Intersections on Smooth Varieties

We begin with a definition.

Definition 2. Let X be a smooth variety with diagonal $\delta : X \to X \times X$. We define the intersection product as the following composition:

$$A_k X \otimes A_\ell X \xrightarrow{\times} A_{k+\ell} X \times X \xrightarrow{\delta^*} A_{k+\ell-n} X$$

More generally, let X be as above and let Y be any scheme with morphism $f : X \to Y$ and graph γ_f. If $\dim X = n$ then we define the cap product

$$f^* \frown : A_i X \otimes A_j Y \to A_{i+j-n} Y \qquad\qquad f^*(x) \frown y = \gamma_f^*(x \times y)$$

Note that Proposition 1 allows us to use these Gysin pullbacks both in the case of the diagonal and in the case of the graph morphism. As we did above, we may define refined intersection products by using the refined maps instead. In this framework, let x, y be cycles on Y with supports $|x|, |y|$. We then have the following fibre square

$$\begin{array}{ccc} |x| \cap |y| & \longrightarrow & |x| \times |y| \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \times Y \end{array}$$

Then with our refined notions from above, we have $\delta^!(x \times y) \in A_*(|x| \cap |y|)$ which then maps into the global chow group A_*Y. This leads to the following definition:

Definition 3. Let $f : X \to Y$ a morphism of schemes with Y a nonsingular dimension n variety. Let $p_X : X' \to X$ and $p_Y : Y' \to Y$ be morphisms with $x \in A_k X'$ and $y \in A_\ell Y'$ and let γ_f denote the graph of f. Then we have the following square

$$\begin{array}{ccc} X' \times_Y Y' & \longrightarrow & X' \times Y' \\ \downarrow & & \downarrow{\scriptstyle p_X \times p_Y} \\ X & \xrightarrow{\gamma_f} & X \times Y \end{array}$$

and we may define

$$x \cdot_f y = \gamma_f^!(x \times y) \in A_{k+\ell-n}(X' \times_Y Y')$$

Note that if $X' = X$ and $Y' = Y$ then this is just what we had before because p_X, p_Y become the identities on X, Y respectively. The behavior of these definitions is exactly as expected, summed up in the proposition below:

Proposition 4. *Let X, Y, Y_i, Z be varieties, $p_X : X' \to X$, $p_Y : Y' \to Y$, $p_{Y_i} : Y_i' \to Y_i$, $p_Z : Z' \to Z$, $x \in A_*X$, $y \in A_*Y$, $z \in A_*Z$, and $y_i \in A_*Y_i$. Then*

1. *(Associativity) If $f : X \to Y$ and $g : Y \to Z$ are morphisms with Y, Z nonsinigular, then*

$$x \cdot_f (y \cdot_g z) = (x \cdot_f y) \cdot_f gfz \in A_*(X' \times_Y Y' \times_Z Z')$$

2. *(Commutativity) If $f_i : X \to Y_i$ with Y_i nonsingular then*

$$(x \cdot_f f_1 y_1) \cdot_{f_2} y_2 = (x \cdot_f f_2 y_2) \cdot_{f_1} y_1 \in A_* Y_1' \times_Y Y_1 X' \times_{Y_2} Y_2'$$

3. *(Projection Formula) Let $f : X \to Y$, $g : Y \to Z$ with Z nonsingular. Suppose $f' : X' \to Y'$ is proper and $p_Y f' = f p_X$. Let $f'' = f' \times_Z id_Z : X' \times Z \to Y' \times Z$. Then*

$$f''_*(x \cdot_{gf} z) = f'_*(x) \cdot_g z \in A_*(Y' \times_Z Z')$$

4. *(Compatibility) Let $f : X \to Y$ with Y nonsingular and let $g : V' \to Y'$ be a regular imbedding. Then we have*

$$g^!(x \cdot_f y) = x \cdot_f g^! y \in A_*(X' \times_Y V')$$

Proof. For (1), consider

$$\begin{tikzcd} X \arrow[r, "\gamma_{gf}"] \arrow[d, "\gamma_f"] & X \times Z \arrow[d, "\gamma_f \times \mathrm{id}_Z"] \\ X \times Y \arrow[r, "\mathrm{id}_X \times \gamma_g"] & X \times Y \times Z \end{tikzcd}$$

We have a map $X' \times Y' \times Z' \to X \times Y \times Z$ which induces another fiber square over this one. Then we have

$$x \cdot_f (y \cdot_g z) = \gamma_f^!(x \times \gamma_g^!(y \times z)) = \gamma_f^!(\mathrm{id}_X \times \gamma_g)^!(x \times y \times z) = (\gamma_f \times \mathrm{id}_Z)^!(\mathrm{id}_X \times \gamma_g)^!(x \times y \times z)$$

by our commutativity theorem from [Ful98, §6]. Now we may just apply the definitions to get

$$(\gamma_f \times \mathrm{id}_Z)^!(\mathrm{id}_X \times \gamma_g)^!(x \times y \times z) = \gamma_{gf}^!((x \cdot_f y) \times z) = (x \cdot_f y) \cdot_{gf} z$$

as desired.

For (2) we do the same thing with the fiber square

$$\begin{tikzcd} X \arrow[r, "\gamma_{f_2}"] \arrow[d, "\gamma_{f_1}"] & X \times Y_2 \arrow[d, "\gamma_{f_1} \times \mathrm{id}"] \\ Y_1 \times X \arrow[r, "\mathrm{id} \times \gamma_{f_2}"] & Y_1 \times X \times Y_2 \end{tikzcd}$$

We note that (3) and (4) are proved similarly, with increasingly complicated diagrams. Details can be found in [Ful98, §8] ■

We have the following useful corollaries.

Corollary 5. *Let $f : X \to Y$ with Y nonsingular. Let x be a cycle on X. Then $x \cdot_f [Y] = x$.*

Proof. By (c) in Proposition 4, we may assume that $x = [X]$ by considering inclusion of our cycle into X and applying the projection formula. Then we have by definition

$$x \cdot_f [Y] = \gamma_f^*[X \times Y] = [\gamma_f^{-1}(X \times Y)] = [X]$$

as desired. ■

Note that Corollary 5 will show that the intersection ring has the expected identity in that obviously if you intersect with a containing space, the intersection class should not change.

Corollary 6. *Suppose Y is nonsingular and let $j : V \to Y$ be a regular imbedding. Let x be a cycle on Y. Then*

$$x \cdot [V] = j^!(x) \in A_*(|x| \cap V)$$

Proof. By Corollary 5 and Proposition 4, we have the following chain of equalities:

$$j^!(x) = j^!(x \cdot_f [Y]) = x \cdot_f j^![Y] = x \cdot_f [V]$$

from the definition of $j^! : A_*Y \to A_*V$. ■

Our last corollary justifies our notion of intersection product.

Corollary 7. *If $f : X \to Y$ is a morphism of nonsingular varieties and $\Gamma \subset X \times Y$ is the graph, then for all cycles $x \in A_*X$ and $y \in A_*Y$ we have*

$$x \cdot_f y = [X \times Y][\Gamma]$$

Proof. Let $\gamma : \Gamma \to X \times Y$ denote the inclusion. Then by Corollary 5, we have

$$(x \times y) \cdot [\Gamma] = \gamma^!(x \times y) = x \cdot_f y$$

by definition. ■

Note that in the special case of $f : X \to X$ being the identity, this shows that

$$x \cdot y = [x \times y] \cdot [\Delta]$$

where Δ is the diagonal. This is, of course, exactly as expected.

Note that Corollary 6 shows that the classic fact of the Euler characteristic being given by the self intersection of the diagonal holds in algebraic geometry even though we do not have a tubular neighborhood theorem. To see this, we first recall

Proposition 8. *Given a fiber diagram*

$$\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{i'} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

with i a regular imbedding and

1. *(Excess Intersection) with i' regular imbeddings of codimensions d and d' and normal bundles N and N', and $E = g^*N/N'$, then $i^! = c_{d-d'}(q^*E) \cap i'^!$ as morphisms $A_*Y'' \to A_{*-d}X''$.*
2. *(Push-forward) with p proper $i^!p_* = q_*i^!$*
3. *(Pull-back) with p flat, $i^!p^* = q^*i^!$.*

Proof. See [Ful98, §§6.2-3] ■

If X is nonsingular and $\Delta \subset X \times X$ is the diagonal, then we have $[\Delta] \cdot [\Delta] = \delta^!(\Delta)$. We note Proposition 8 contains the special case of q and p and i' isomorphisms, which then works out to $i^! = c_d(g^*N)\cap : A_*Y' \to A_{*-d}X'$. Going back to our example, we have

$$\begin{array}{ccc} \Delta & \longrightarrow & \Delta \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle i} \\ X & \xrightarrow{\delta} & X \times X \end{array}$$

and $\delta^!(\Delta) = c_n(g^* N_\Delta X \times X) \cap [\Delta]$. But we have already seen that this is given by $c_n(T_X) \cap [X]$.

With our work with refined intersections above, we may define a refined Gysin map as follows.

Definition 9. Let $f : X \to Y$ be a morphism from a purely m-dimensional scheme X to a nonsingular n-dimensional variety Y. Let $g : Y' \to Y$ be a morphism and define $X' = X \times_Y Y'$. Then we have the refined Gysin map

$$f^! : A_k Y' \to A_{k+m-n} X' \qquad\qquad y \mapsto [X] \cdot_f y$$

The key result regarding this refined Gysin map is that it does not really give us anything new, as encapsulated in the following proposition.

Proposition 10. *Let $f : X \to Y$ a morphism with X, Y as in the definition above. Then, if f is flat and $f' : X' \to Y'$ is the induced morphism, then $f^! = f'^*$.*

Proof. This is immediate from the following compatibility result found in [Ful98, §6.5]. Given the commuting diagram

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & Z' \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle \varphi} \\ X & \xrightarrow{i} & Y & \xrightarrow{p} & Z \end{array}$$

where i is a regular imbedding of codimension d and p is flat of relative dimension n, and $p \circ i$ is flat of relative dimension $n - d$, then i' is a regular imbedding of codimension d, p' is flat as well and $(p'i')^* = i'^* p'^* = i^! p'^*$. Let $Z = Y$ and let $p = \mathrm{id}$ and $i = f$. Then we have

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \xrightarrow{\mathrm{id}} & Y' \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle \varphi} \\ X & \xrightarrow{f} & Y & \xrightarrow{\mathrm{id}} & Y \end{array}$$

and we have

$$f^! = f^! \mathrm{id}^* = (\mathrm{id} \circ f')^* = f'^*$$

as desired. ∎

This compatibility result is useful in that it gives us the classic projection. Let $i : V \to Y$ be a regular imbedding into a nonsingular variety. Then for all $y \in A_* Y$, we have $[V] \cdot y = i_* i^*(y)$. This follows because $i^*(y) = i^!(y) = [V] \cdot y \in A_*(V \cap |y|)$ by Proposition 10 and Corollary 6. Pushing forward yields the projection.

3 The Intersection Ring and Examples

With the above setup, and guided by intuition from the case of real manifolds and Poincaré duality, we give the Chow groups a natural ring structure for nonsingular varieties.

Definition 11. Let Y be an n-dimensional nonsingular variety. We define $A^pY := A_{n-p}Y$. The intersection product gives maps

$$A^pY \otimes A^qY \to A_{n-p+n-q-n}Y = A_{n-p-q}Y = A^{p+q}Y$$

and similarly if $f : X \to Y$ is a morphism then the cap product gives

$$A^pY \otimes A_qX \xrightarrow{\frown} A_{q-p}X$$

Note that if X is m dimensional nonsingular and f is flat then

$$f^* : A^pY = A_{n-p}Y \to A_{n-p+m-n}X = A^pX$$

and so f^* preserves degrees. We have the following proposition.

Proposition 12. *Let Y be a nonsingular n-dimensional variety. Then*

1. *As defined above A^*Y is an associative, commutative ring with unit given by $[Y]$. Moreover, this assignment is functorial from nonsingular varieties to rings with flat morphisms f being sent to their pullbacks f^*.*
2. *If $f : X \to Y$ is a morphism from some shceme X then the cap product turns A_*X into an A^*Y-module.*
3. *If $f : X \to Y$ is proper and X is a nonsingular variety, then for all $x \in A^*X$ and $y \in A^*Y$, we have*

$$f_*(f^*y \cdot x) = y \cdot f_*(x)$$

Proof. We have that associativity and commutativity follow from (1) and (2) in Proposition 4 and unit follows from Corollary 5. Functoriality follows as well from (1) in Proposition 4 and (3) above follows from (3) in Proposition 4. The fact that A_*X is an A^*Y-module similarly follows immediately from Proposition 4 and the definition of cap product. ■

Example 13. A natural question is how might we define such a ring in the more general case where we may drop nonsingularity or irreducibility. Thus we may define for some quasi-projective X,

$$A^*X = \varinjlim A^*Y$$

where the colimit is over all pairs (Y, f) such that $f : X \to Y$ is a morphism and Y is a nonsingular quasi-projective variety. Note that this is clearly contravariant as a functor, and our notions of cap products, the projection formula, and chern classes all hold. This notion also reduces to our above notion in the case of X nonsingular because it is clear that if X is nonsingular, then X is inital in the category of nonsingular varieties with morphisms from X. Unfortunately, in most cases it is very difficult to understand this ring explicitly; thus the notion of bivariant intersection theory and operational chow groups is introduced in [Ful98, §17].

We now move on to consider intersections in projective space. We recall from the first lecture that, using affine stratifications, we have

$$A_k\mathbb{P}^n = \begin{cases} \mathbb{Z} & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

with $A_k\mathbb{P}^n$ being generated by the class of a k-plane, $[L_k]$. From the fact that if V, W are transverse then $[V] \cdot [W] = [V \cap W]$, we see immediately that if H^i is a codimension i plane then $[H^i][H^j] = [H^{i+j}]$ with $[H^{i+j}] = 0$ if $i + j > n$. This is summed up in the following result:

Proposition 14. *If $\mathbb{P}^n$ is projective space then $A^*\mathbb{P}^n = \mathbb{Z}[\zeta]/(\zeta^{n+1})$.*

We have the following examples.

Example 15. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. and let $i : X \to \mathbb{P}^3$ be the Segre imbedding, $i((x_0 : x_1), (y_0 : y_1)) = (x_0y_0 : x0y_1 : x_1y_0 : x_1y_1)$. We know $A^*X = \mathbb{Z}[x_1, x_2]/(x_1^2, x_2^2)$. What is $i^* : A^1\mathbb{P}^3 \to A^1X$? We know that $A^1\mathbb{P}^3 = \mathbb{Z} \cdot [H]$ where H is a hyperplane. Let $H : x_3 = 0$. Then we have

$$i^*[H] = [i^{-1}(H)] = [(1:0) \times (y_0 : y_1) \cup (x_0 : x_1) \times (1:0)] = [\{pt\} \times \mathbb{P}^1] + [\mathbb{P}^1 \times \{pt\}] = x_1 + x_2$$

Thus, if we let the bidgree of $\alpha = ax_1 + bx_2 \in A^1X$ be (a, b), then $\operatorname{Im} i^*$ is the set of elements of bidegree (a, a). If $Y \subset \mathbb{P}^3$ is a degree d surface, then $i^*[Y] = (d, d)$. Thus if we have C an irreducible closed curve on X of bidegree (a, b) with $a \neq b$, then we cannot realize C as an intersection of X with any surface in $\mathbb{P}^3$.

Example 16. We have seen that if $X = \mathbb{P}^m \times \mathbb{P}^n$ and s, t represent the classes of hyperplanes in pp^m, $\mathbb{P}^n$ respectively, we have $A^*X = \mathbb{Z}[s, t]/(s^{m+1}, t^{n+1})$. Thus if $V \subset X$ is a subvariety of dimension k then we have

$$[V] = \sum_{i+j=k} a_{ij} s^{m-i} t^{n-j}$$

with the a_{ij} called the bidegrees of V. We wish to turn these bidegrees into a degree. We may do this by the following process. Consider the ideal $\mathfrak{p} \subset K[X_0, \ldots X_m, Y_0, \ldots Y_n]$ that is homogeneous in X and in Y defining V. Let $\mathfrak{p}' \subset \mathfrak{p}$ be the subset of those elements that are homogeneous in all of the variables and is not just bihomogeneous. Then let V' be the variety corresponding to $\mathfrak{p}'$. This can be realized geometrically set theoretically as

$$V' = \{(\lambda x_0 : \lambda x_1 : \ldots \lambda x_m : \mu y_0 : \cdots : \mu y_n) \in \mathbb{P}^{m+n+1} | (x, y) \in V, \, (\lambda : \mu) \in \mathbb{P}^1\}$$

Then it is a classical result that the degree of V' is the sum of the bidegrees of V.

We now focus entirely on intersections in projective space. Because of Proposition 14, we know that the intersection ring is particularly well-behaved, and thus we can see explicit and concrete results relatively easily. For $\alpha \in A_k\mathbb{P}^n$ we define $\deg \alpha = \int_{\mathbb{P}^n} c_1(\mathcal{O}(1))^k \cap \alpha$, constructed such that $\alpha = \deg(\alpha) \cdot \zeta^{n-k}$ where ζ is the class of a hyperplane. From Proposition 14 above, we immediately get

Proposition 17 (Bézout)**.** *Let $\alpha_i \in A^{\alpha_i}\mathbb{P}^n$. If $\sum \alpha_i \leq n$ then* $\deg(\alpha_1 \ldots \alpha_r) = \deg \alpha_1 \deg \alpha_2 \ldots \deg \alpha_r$.

Example 18. A natural question to ask is if we can apply this degree in the Bezout sense to the irreducible components of an intersection in a way such that the sum of these numbers give the degree of the intersection. The answer to this question is no if we want these numbers to be preserved by automorphisms of $\mathbb{P}^n$ as we see in the following example. Let $n = 4$ and consider

$$V = V(x_3^3 - x_1x_2(x_2 - 2x_1), x_3)$$
$$W = V(x_4^3 - x_2x_1(x_1 - 2x_2), x_4)$$

Then $\deg V = \deg W = 3$ so $\deg V \cap W = 9$. Note that $V \cap W$ is given by $x_3 = x_4 = 0$ and at least one of x_1, x_2 also zero. Thus $V \cap W$ is the union of lines L_1, L_2 where

$$L_1 : x_1 = x_3 = x_4 = 0$$
$$L_2 : x_2 = x_3 = x_4 = 0$$

Let $\sigma \in S_4$ such that $\sigma = (1\,2)(3\,4)$. Then σ acting on indices is an automorphism of $\mathbb{P}^4$ and $\sigma \cdot V = W$, $\sigma \cdot W = V$, $\sigma \cdot L_1 = L_2$ and $\sigma \cdot L_2 = L_1$. Thus suppose we assing numbers n_1, n_2 to L_1, L_2. If this assignment of numbers is invariant under $\operatorname{Aut} \mathbb{P}^4$ then it is invariant under σ and so $n_1 = n_2$. But we then have $9 = 2n_1$ and so $n_1 \notin \mathbb{Z}$.

As an example application, we will use the developed theory to prove the following claim

Claim 19. *Let $X \subset \mathbb{P}^n$ be an irreducible subvariety of degree d and suppose that X is not contained in any hyperplane of $\mathbb{P}^n$. Then*

$$\dim X + d \geq n + 1$$

Before we do this, we make the following claim:

Claim 20. *All intersection products in $\mathbb{P}^n$ can be realized as a subvariety intersecting a linear subspace in some $\mathbb{P}^N$.*

To prove Claim 20, we introduce the notion of a ruled join. Let $V, W \subset \mathbb{P}^n$ be varieties of dimensions k, ℓ respectively. We define $J = J(V,W) \subset \mathbb{P}^{2n+1}$ as follows. Let $\mathbb{P}^n_1, \mathbb{P}^n_2 \subset \mathbb{P}^{2n+1}$ be given by $x_{n+1} = \cdots = x_{2n} = 0$ and $x_0 = x_1 = \cdots = x_n = 0$ respectively. Consider $V \subset \mathbb{P}^n_1$ and $W \subset \mathbb{P}^n_2$ and define J as the union of lines in $\mathbb{P}^{2n+1}$ joining a point in V and a point in W. If $\mathfrak{p}, \mathfrak{q}$ are the ideals defining V, W respectivley, then the ideal of J is given by $(\mathfrak{p}, \mathfrak{q}) = \mathfrak{a} \subset k[x_0, \ldots, x_n, y_0, \ldots, y_n]$. Let $L \subset \mathbb{P}^{2n+1}$ be given by all points $(x : y)$ such that $x = y$ and note that $i : \mathbb{P}^n \to \mathbb{P}^{2n+1}$ given by $x \mapsto (x : x)$ induces an isomorphism $i : \mathbb{P}^n \xrightarrow{\sim} L$. Thus we have

$$i : V \cap W \xrightarrow{\sim} L \cap J$$

because clearly if $u = (x : y) \in L \cap J$ then $x \in V$, $y \in W$ and $x = y$. Define $\mathbb{P}^{2n+1}_\circ = \mathbb{P}^{2n+1} \setminus (\mathbb{P}^n_1 \cup \mathbb{P}^n_2)$ and let $\pi : \mathbb{P}^{2n+1}_\circ \to \mathbb{P}^n \times \mathbb{P}^n$ be given by $(x : y) \mapsto x \times y$. Note that this is well defined because we do not have either $x = 0$ or $y = 0$ in $\mathbb{P}^{2n+1}_\circ$. Note that $\pi : L \xrightarrow{\sim} \Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ and $\pi : L \cap J \xrightarrow{\sim} \Delta \cap (V \times W)$. We know that π is smooth. We claim that $V \cdot W = L \cdot J \in A_{k+\ell-n}(V \cap W)$. To see this, let $J_\circ = \mathbb{P}^{2n+1}_\circ \cap J$ and consider

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "i_\circ"] \arrow[d, "\mathrm{id}"] & \mathbb{P}^{2n+1}_\circ \arrow[d, "\pi"] & J_\circ \arrow[l, hook'] \arrow[d, "\pi'"] \\
\mathbb{P}^n \arrow[r, "\delta"] & \mathbb{P}^n \times \mathbb{P}^n & V \times W \arrow[l, hook']
\end{tikzcd}$$

where $i_\circ : \mathbb{P}^n \to \mathbb{P}^{2n+1}_\circ$ is just $i : \mathbb{P}^n \to L \subset \mathbb{P}^{2n+1}_\circ$. Then we have

$$V \cdot W = \delta^![V \times W] = i_\circ^! \pi'^*[V \times W] = i_\circ^![\pi'^{-1}(V \times W)] = i_\circ^![J_\circ]$$

Then considering Proposition 8, we have

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "i_\circ"] \arrow[d, "\mathrm{id}"] & \mathbb{P}^{2n+1}_\circ \arrow[d] \\
\mathbb{P}^n \arrow[r, "i"] & \mathbb{P}^{2n+1}
\end{tikzcd}$$

We know that $[J_\circ] \in A_* \mathbb{P}^{2n+1}_\circ$ and so $i_\circ^![J_\circ] = i^![J]$. We may then apply Corollary 6 to have $i^![J] = L \cdot J$. Thus we have $V \cdot W = L \cdot J(V,W)$. We see that L is a linear subspace so Claim 20 holds. We remark that taking degrees and recalling that linear spaces have degree one, we have that $\deg J(V,W) = \deg V \deg W$. We might adapt this argument to consider r intersections in projective space, which would yield the ruled join in $\mathbb{P}^{r(n+1)-1}$. Thus, any intersection that occurs in projective space may be realized as the intersection of a variety and some linear subspace.

The preceding discussion allows us to make the following claim, a common corollary of Bezout's theorem.

Claim 21. *If $V_1, \ldots V_r \subset \mathbb{P}^n$ and $Z_1, \ldots Z_s$ are the irreducible components of $\bigcap_i V_i$, then*

$$\sum \deg(Z_i) \leq \prod \deg(V_i)$$

By induction, we may take $r = 2$ and by Claim 20, we may assume that V_2 is linear. Thus we are considering $V \cap L$. But L is the intersection of hyperplanes, so again by induction we may consider $V \cap H$ where H is a hyperplane. If $V \subset H$ then $V \cap H = V$ and $Z = V$ a single irreducible component; then equality holds. If $V \not\subset H$ then $V \cdot H = \sum a_i [Z_i]$ with each $a_i \geq 1$. Thus

$$\deg V \deg H = \deg(\sum a_i [Z_i]) = \sum a_i \deg Z_i \geq \sum \deg Z_i$$

An easy application of this is

Example 22. If X is projective and $V_1, \ldots, V_r \subset X$ are subvarieties of degrees $d_1, \ldots d_r$ with respect to a given imbedding of the subvarieties, then if $\bigcap V_i$ is finite then

$$\left|\bigcap V_i\right| \leq d_1 \cdots d_r$$

Now we are finally prepared to prove Claim 19. Indeed, Let $X \subset \mathbb{P}^n$ be a subvariety of degree d not contained in any hyperplane. Taking a generic hyperplane $H \cong \mathbb{P}^{n-1}$ we may consider $X \cap H \subset \mathbb{P}^{n-1}$. If H is general, then by a variant of Bertini's theorem, we may take $X' = X \cap H$ is a $\dim X - 1$ irreducible subvariety not contained in any hyperplane in $\mathbb{P}^{n-1}$. Inducting, it suffices to consider the case X a curve of degree d imbedded in $\mathbb{P}^n$ not contained in any hyperplane. We need to show that $d \geq n$. Choose n points on X. Then there is a hyperplane H through those n points and so $d = \deg X \deg H \geq \sum \deg Z_i \geq n$. Thus Claim 19 is established.

References

[Ful98] William Fulton. *Intersection Theory, 2nd Edition.* Springer, 1998.