Basic Theory of Finite Group Representations over $\mathbb{C}$

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1 Introduction

We introduce the basics of the representation theory of finite groups in characteristic zero. In the sequel, all groups G will be finite and all vector spaces V will be finite dimensional over $\mathbb{C}$. We first define representations and give some basic examples. Then we discuss morphisms and subrepresentations, followed by basic operations on representations and character theory. We conclude with induced and restricted representations and mention Frobenius Reciprocity. These notes are intended as the background to the UMS summer seminar on the representation theory of symmetric groups from [AMV04]. The author learned this material originally from [FH] and recommends this source for more detail in the following; many of the proofs in the sequel likely come from this source.

2 Definitions and Examples

Representation theory is the study of groups acting on vector spaces. As such, we have the following definition:

Definition 1. A representation of a group G is a pair (V, ρ) where V is a vector space over $\mathbb{C}$ and ρ is a homomorphism $\rho : G \to \mathrm{GL}(V)$. We will often refer to representations by their vector space and assume that the morphism ρ is clear from context. Every representation defines a unique $\mathbb{C}[G]$-module, with the action of G on V being $g \cdot v = \rho(g)(v)$ and vice versa. We refer to the dimension of the representation, defined to be $\dim V$.

With the definition in mind, we begin with a few examples.

Example 2. Let G be any group and V any vector space. Let ρ be the identity morphism. This defines a representation When $\dim V = 1$ we call this the *trivial representation*.

Example 3. Let G be a finite group and let

$$V = \bigoplus_{g \in G} \mathbb{C}g$$

the vector space generated by elements $g \in G$. There is a natual action of G on V given by $g \cdot h = (gh)$ with multiplication in the group G. This defines a representation that we refer to as the *regular representation*.

Example 4. Let $G = S_n$ be the symmetric group and let $V = \mathbb{C}$ be one dimensional. Recall that there is a homomorphism $\varepsilon : S_n \to \{\pm 1\}$ called the sign homomorphism. We may define a representation $g \cdot v = \varepsilon(g)v$ that we call the *alternating representation*.

Example 5. Let $G = S_n$ and let V be an n-dimensional vector space with basis $v_1, \ldots, v_n$. Define the action $\sigma \cdot v_i = v_{\sigma^{-1}(i)}$. This representation is called the *standard representation*.

Now that we have a definition of what a representation is, we may wish to make a nice category. For this we need morhpisms.

Definition 6. Given two representations $(V, \rho), (V', \rho')$ of a group G, we define a morphism φ of representations as a linear map $\varphi : V \to V'$ that commutes with the action of G, in that $\rho(g)(\varphi(v)) = \varphi(\rho(g)(v))$. In other words, the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ V' & \xrightarrow{\rho'(g)} & V' \end{array}$$

An isomorphism is a bijective morphism. Given two representations we let $\mathrm{Hom}(V, W) = \mathrm{Hom}_G(V, W)$ be the set of morphisms (G-equivariant linear maps) from V to W. Note that it has the structure of a vector space.

Clearly the composition of morphisms is again a morphism. Now that we have the concept of isomorphism, when we refer to a representation we will generally be referring only to a representation up to isomorphism.

Now there are several important functorial constructions associated with a representation. The first is the direct sum. Let V and W be representations of G. Then we may define the representation $V \oplus W$ with action $g \cdot (v, w) = (g \cdot v, g \cdot w)$ with $\dim V \oplus W = \dim V + \dim W$ We may form the tensor product $V \otimes W$ with associated action $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$ and the dual representation V^* with associated action $\rho^*(g) = \rho^t(g^{-1})$. Symmetric and exterior powers of representations may be defined by quotienting subspaces of tensor powers; it is left to the reader to check that these subspaces are subrepresentations. We recall $\mathrm{Hom}(V, W) = V^* \otimes W$ (if you do not know this, prove it!) and so we may define a representation structure on $\mathrm{Hom}(V, W)$. We now need the concept of a subrepresentation.

Definition 7. Let V be a representation of G. A subspace $W \subset V$ is a subrepresentation if it is stable under the action of G, i.e., $G \cdot W = W$. We call a representation simple (or irreducible) if it has no nontrivial subrepresentations.

Example 8. Let V be any representation of G. Then the subspace $W = \{0\}$ is stable under the action of G and so $W \subset V$ is a subrepresentation. Similarly, $V \subset V$ and is clearly stable under the action of G so this is also a subrepresentation. For any morphism $\varphi : V \to W$, note that $\ker \varphi \subset V$ is a subrepresentation and similarly for $\mathrm{Im}\, \varphi \subset W$.

Example 9. Let G be any group and let V be a two dimensional representation with trivial G-action. Then any subspace of V is a subrepresentation.

Example 10. Let $G = S_3$ and let V be the standard representation of Example 5. Then consider the space $W = \mathbb{C} \cdot (v_1 + v_2 + v_3)$. This is a 1-dimensional subrepresentation.

Given a subrepresentation, a natural question to ask is how this subrepresentation splits off. The answer to this is as nice as we could possibly want:

Proposition 11. *Let $W \subset V$ be a sub representation. Then there is another subrepresentation $W' \subset V$ such that $V = W \oplus W'$ splits as representations of G.*

Proof. Note first that if there is a G-equivariant inner product, $\langle \cdot, \cdot \rangle$, then we may define

$$W^\perp = \{v \in V | \langle w, v \rangle = 0, \text{ for all } w \in W\}$$

Then $V = W \oplus W^\perp$. Moreover, for any $v \in W^\perp$, $g \in G$, we have for all $w \in W$,

$$0 = \langle v, w \rangle = \langle g \cdot v, g \cdot w \rangle = \langle g \cdot v, g \cdot (g^{-1} w') \rangle = \langle g \cdot v, w' \rangle$$

where $w' = g \cdot w \in W$ by assumption of W being a subrepresentation. Thus $g \cdot v \in W^\perp$ and so we may set $W' = W^\perp$. Now, choose an arbitrary inner product $\langle \cdot, \cdot \rangle_0$ on V and define

$$\langle v, v' \rangle = \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot v' \rangle_0$$

Then it is clear that $\langle \cdot, \cdot \rangle$ is a G-equivariant inner product and we are done. ∎

Example 12. Consider the situation of Example 8. Then the complementary subrepresentation W' is just $W' = \{av_1 + bv_2 + cv_3 | a + b + c = 0\}$. To see this, we may introduce the inner product $\langle v_i, v_j \rangle = \delta_{ij}$ and compute to see that it is G-equivariant.

By applying descent on the dimension of a representation, Proposition 11 immediately yields that all representations are direct sums of simple representations and so it suffices to study simple representations.

Remark 13. Note that the conclusion of Proposition 11 is manifestly false in general if either $\operatorname{char} k > 0$ or G is infinite, although if G is compact and $k = \mathbb{C}$ then many of the results hold. Fortunately, since the goal of these notes is to prepare for [AMV04], these issues are immaterial here.

One of the important results that governs the morphisms between representations is the classical Schur's lemma.

Proposition 14 (Schur's Lemma). *Let V, W be simple representations of G and let $\varphi : V \to W$. Then φ is either 0 or an isomorphism. Moreover, if $V = W$ then φ is just an action by scalars. Thus we have*

$$\operatorname{Hom}(V, W) = \begin{cases} \mathbb{C} & V \cong W \\ 0 & V \ncong W \end{cases}$$

Thus every representation can be uniquely expressed as a direct sum of simple representations.

Proof. Recall from Example 8 that $\ker \varphi \subset V$ is a subrepresentation of V. Because V is simple, we have either $\ker \varphi = V$ or $\ker \varphi = 0$. Similarly $\operatorname{Im} \varphi \subset W$ and so $\operatorname{Im} \varphi = 0$ or $\operatorname{Im} \varphi = W$. The first statement follows immediately. Now suppose $\varphi : V \to V$ is a morphism. Then because $\mathbb{C}$ is algebraically closed, φ must have an eigenvalue λ. But then we have $\varphi - \lambda \cdot \mathrm{id} : V \to V$ has nonzero kernal and so by the above result this must be the zero map. Thus $\varphi = \lambda \cdot \mathrm{id}$. The penultimate statement follows immediately, as does the last statement. ∎

Example 15. We may use Proposition 14 to classify all simple representations of an abelian group G. Suppose V is a simple representation of G abelian and for some $g \in G$, let $\varphi : V \to V$ be $v \mapsto g \cdot v$. Then φ is linear by definition and for any $g' \in G$ we have $g' \cdot \varphi(v) = (g'g) \cdot v = (gg') \cdot v = \varphi(g' \cdot v)$ and so φ is a morphism of representations. Thus by Proposition 14, we have that $\varphi(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. But this then means that if $W \subset V$ is any *subspace* then it is stable under the action of G and so it is indeed a *subrepresentation*. Thus if V is simple then $\dim V = 1$.

3 Restriction and Induction

Recall that following the definition of a morphism, there were a few functorial constructions described. Each of these had a fixed *group* and the *representation* changed. A natural question is what to do if one wishes to fix instead the representation and change the group. There are two functorial constructions that do this.

Definition 16. Let $H < G$ be a subgroup and let V be a representation of G. Then we define the restriction of V to H, $\operatorname{Res}_H^G V$ to be the representation $(V, \rho|_H)$. Note that we will omit one or both of the groups from the notation if context makes such redundant.

Example 17. This concept is pretty intuitive, but one important fact to note is that the restriction of a simple representation is no longer necessarily simple. Indeed, recall from Example 12 that we identified $W' = \{av_1 + bv_2 + cv_3 | a + b + c = 0\}$ as a (two dimensional) representation of S_3. It is easy to check by hand (or using the results of § 4) that this representation is simple. Now, let $H = S_2$ identified in the normal way as a subgroup of S_3. Then $U = \operatorname{Res}_H W'$ is manifestly *not* simple. Indeed S_2 fixes v_3 so the space spanned by $v_1 + v_2 - 2v_3$ is stable under the action of S_2 and so is a subrepresentation.

The complement to restriction is induction:

Definition 18. Let $H < G$ and let V be a representation of H. We define

$$\operatorname{Ind}_H^G V := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$$

Equivalently, if R is a set of representatives of G/H then we define

$$\operatorname{Ind}_H^G = \bigoplus_{s \in R} sV$$

with the obvious action of $g \cdot (sw) = s' \cdot (hw)$ where $gs = s'h$ and $s' \in R$. Note that we will omit the groups from the notation if the context makes such redundant.

Example 19. There is an action of G on R which leads naturally to a representation of G, say W. Then it is immediate that $W = \operatorname{Ind}_H^G V$ where V is the trivial representation of H. An extreme example of this is the regular representation from Example 3 where we just let $H = \{1\}$. Then $\operatorname{Ind}^G V$ is exactly the regular representation.

It is easy to see that both of these operations are transitive, i.e., if we have $K < H < G$ are groups and V is a representation of K and W is a representation of G, then

$$\operatorname{Ind}_H^G(\operatorname{Ind}_K^H V) = \operatorname{Ind}_K^G V \qquad \text{and} \qquad \operatorname{Res}_K^H(\operatorname{Res}_H^G W) = \operatorname{Res}_K^G W$$

While this fact is immediate, it might be a good idea to check your understanding by showing this. When we called Res and Ind complementary, we meant that they are adjoint functors in the following sense:

Proposition 20 (Frobenius Reciprocity I)**.** *Let $H < G$ be groups, W a representation of H and V a representation of G. Then*

$$\operatorname{Hom}_H(W, \operatorname{Res} V) = \operatorname{Hom}_G(\operatorname{Ind} W, V)$$

Proof. Let

$$V = \bigoplus_{s \in R} s \cdot W$$

and let $\varphi : W \to \operatorname{Res} U$. Now define $\widetilde{\varphi}$ by the following composition

$$s \cdot W \xrightarrow{s^{-1}} W \xrightarrow{\varphi} V \xrightarrow{s} V$$

Note that because φ is H-equivariant, this definition is independent of choice of R. The inverse construction takes $\psi : \operatorname{Ind} W \to U$ and restricts to $\psi|_W$. These are clearly inverses so we are done. ∎

Another version of this statement will be seen in the following section.

4 Character Theory

While Proposition 11 allows us to consider only irreducible representations, it can be difficult to tell if a representation is simple just by staring at it. Moreover, we might wonder how many irreducible representations there are and how we might find them all. These questions are all answered by character theory.

Definition 21. Given a group G and a representation V we define the character, $\chi_V : G \to \mathbb{C}$ defined by $\chi_V(g) = \operatorname{Tr}(\rho(g))$.

Example 22. The character of the trivial representation is (unsurprisingly) the trivial character taking $\chi(g) = 1$ for all $g \in G$. If G is the alternating representation of $G = S_n$ then $\chi(\sigma) = \pm 1$ depending on the sign of the permutation.

Example 23. For any representation V, we have $\chi_V(1) = \dim V$ by definition. Now let V be the regular representation for some group G and $g \neq 1 \in G$. Considering g as a matrix with respect to the basis given by elements of G, we see that for any h in this basis, $g \cdot h = gh \neq h$. Thus the main diagonal of g is all zeroes and so has trace 0. Thus

$$\chi_V(g) = \begin{cases} |G| & g = 1 \\ 0 & g \neq 1 \end{cases}$$

for V the regular representation.

One of the reasons we use characters is that they behave well with respect to functorial transformation of represntations. We have $\chi_{V\oplus W} = \chi_V + \chi_W$, $\chi_{V\otimes W} = \chi_v \chi_W$ and $\chi^*_{V\oplus W} = \overline{\chi}_V$. These are easy to show by considering what happens to the eigenvalues of g in each of these transformations. Another nice thing about characters is that they are *class functions*, i.e., they are constant on conjugacy classes of G, a fact that follows immediately from the fact that the trace of a linear map is independent of basis.

Now, let $V^G = \{v \in V | g \cdot v = v \text{ for all } g \in G\}$ be the space of invariants. Then consider the map

$$\varphi(v) = \frac{1}{|G|} \sum_{g\in G} g \cdot v$$

that projects $V \to V^G$ (i.e., $\varphi^2 = \varphi$ and $\operatorname{Im}\varphi = V^G$). Then we have

$$\dim V^G = \operatorname{Tr}\varphi = \frac{1}{|G|} \sum_{g\in G} \chi_V(g)$$

by linearity of trace. Now we note that $\operatorname{Hom}(V, W)^G$ is just the space of G-equivariant linear maps, or G-morphisms and we have $\chi_{\operatorname{Hom}(V,W)} = \chi_{V^*\otimes W} = \overline{\chi}_V \chi_W$ by above. Now if we take V, W simple, then we may apply Proposition 14 to get

$$\frac{1}{|G|} \sum_{g\in G} \overline{\chi}_V(g)\chi_W(g) = \begin{cases} 1 & V \cong W \\ 0 & \text{otherwise} \end{cases}$$

This idea leads us to define an inner product on the space of class functions of G by

$$(\chi, \psi) = \frac{1}{|G|} \sum_{g\in G} \overline{\chi}(g)\psi(g)$$

The above ideas lead to the following omnibus result:

Proposition 24. *Let G be a group. With respect to the above inner product, the characters of the irreducible representations are orthonormal. A representation V of G is uniquely determined by its character χ_V and the multiplicity of some irreducible representation W in V is (χ_V, χ_W). Thus $(\chi_V, \chi_V) = \sum a_i^2$ and so a representation is simple if and only if $(\chi_V, \chi_V) = 1$.*

Proof. The first statement follows from above. The second statement follows from the third and Proposition 11. Let $V = \oplus W_i^{\oplus a_i}$ be the decomposition of V into irreducible representations. Then $\chi_V = \sum a_i \chi_i$ and so we have

$$(\chi_V, \chi_{W_i}) = \sum a_i(\chi_{W_j}, \chi_{W_i}) = a_i$$

by orthonormality. Thus we have

$$(\chi_V, \chi_V) = \sum_{i,j} (a_i \chi_{W_i}, a_j \chi_{W_j}) = \sum a_i^2$$

The last statement follows because V is simple if and only if there is a unique $a_i \neq 0$ and then we must have $a_i = 1$. ∎

Example 25. We apply Proposition 24 to the regular representation V of Example 3. Recalling Example 23, we know χ_V. Thus we note by Proposition 24 that

$$(\chi_V, \chi_{W_i}) = \frac{1}{|G|}(|G| \cdot \chi_{W_i}(1)) = \dim W_i$$

and so the regular representation contains all of the irreducible representations with multiplicity the dimension of the representation.

Now we will use the above result to count the irreducible representations of a group G but first we need a lemma.

Lemma 26. *For $\alpha : G \to \mathbb{C}$ any function define for any representation V*

$$\varphi_\alpha = \sum \alpha(g) g : V \to V$$

Then φ_α is a morphism of representations if α is a class function.

Proof. Let $h \in G$ arbitrary. Then

$$\begin{aligned}\varphi_\alpha(h \cdot v) = \sum_{g \in G} \alpha(g) g(h \cdot v) &= \sum_{hgh^{-1} \in G} \alpha(hgh^{-1}) hgh^{-1}(h \cdot v) \\ &= h \sum \alpha(hgh^{-1}) g \cdot v\end{aligned}$$

and

$$h \cdot \varphi_\alpha(v) = h \cdot \sum_{g \in G} \alpha(g) g \cdot v$$

The above are equal if $\alpha(g) = \alpha(hgh^{-1})$ if and only if α is a class function. ∎

Proposition 27. *Given a group G, the set of characters $\{\chi_W\}$ ranging over all irreducible representations W is an orthonormal basis for the space of class functions on G.*

Proof. We have shown that $\{\chi_W\}$ is orthonormal so it suffices to show that they span the class functions, or that any class function α such that $(\alpha, \chi_W) = 0$ for all W satisfies $\alpha = 0$. Consider $\varphi_\alpha : W \to W$ for W irreducible defined in Lemma 26. By Proposition 14, we have that $\varphi_\alpha = \lambda \cdot \mathrm{id}$ and we have

$$\lambda = \frac{1}{\dim V} \operatorname{Tr} \varphi_\alpha = \frac{1}{\dim V} \sum_{g \in G} \alpha(g) \chi_W(g) = \frac{|G|}{\dim V} \overline{(\alpha, \chi_{W^*})} = 0$$

Thus $\varphi_\alpha = 0$ on all representations V. Now let V be the regular representation from Example 3. Then we have

$$0 = \varphi_\alpha(1) = \sum \alpha(g) g(1) = \sum \alpha(g) g$$

but the regular representation is defined such that the g are independent so $\alpha(g) = 0$ for all $g \in G$ and so $\alpha = 0$. ∎

Corollary 28. *The number of irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof. By Proposition 24, the number of irreducible representations of G is the dimension of the space of characters of G. By Proposition 27, this space is the space of class functions of G, which has dimension the number of conjugacy classes of G, compute by taking a basis dual to the set of conjugacy classes of G. ∎

Example 29. Let us consider the example of S_3. Recall that conjugacy classes in S_n correspond bijectively to integer partitions of n. The partitions of 3 are (3), $(2, 1)$, and $(1, 1, 1)$ which means we expect three irreducible representations of S_3. We have the trivial and alternating examples already. Computing the character of W' from Example 12, we see immediately that $(\chi_{W'}, \chi_{W'}) = 1$ and so, by Proposition 24 it is the last irreducible representation.

We conclude with Frobenius reciprocity. Recall from Proposition 20 that we were able to relate induced and restricted representations. Because characters determine everything about a representation, we might expect a similar relation to hold with respect to characters. We are not disappointed.

Proposition 30 (Frobenius Reciprocity II)**.** *Let $H < G$ be groups and define $(\cdot, \cdot)_H$, $(\cdot, \cdot)_G$ to be the inner products on class functions of H, G respectively. Let V be a representation of H and let W be a representation of G. Then we have*

$$(\chi_{\operatorname{Ind} W}, \chi_V)_G = (\chi_W, \chi_{\operatorname{Res} V})_H$$

Proof. By linearity, it suffices to assume that W, V are simple. Then $(\chi_W, \chi_{\operatorname{Res} V})_H$ is the multiplicity of W in $\operatorname{Res} V$, which is just $\dim \operatorname{Hom}(W, \operatorname{Res} V)$ and $(\chi_{\operatorname{Ind} W}, \chi_V)_G$ is the multiplicity of V in $\operatorname{Ind} W$, which is $\dim \operatorname{Hom}(V, \operatorname{Ind} W)$. But by Proposition 20, we have $\operatorname{Hom}(W, \operatorname{Res} V) = \operatorname{Hom}(V, \operatorname{Ind} W)$ so we are done. ∎

References

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