Artin L-Functions

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1 Introduction

Artin L-functions originated with an analytic question relating to the Dedekind ζ function. Where before, L-series were formed from functions on a number field, Artin generalized his series to be based upon a fixed representation of the Galois group of an extension. Of course, in the abelian case, class field theory gives a fundamental connection between the two foundations, and Artin L-series are now being used in an attempt to create a nonabelian class field theory. We begin by stating some necessary preliminary results regarding the representation theory of finite groups and certain basic notions from algebraic number theory. We then define the Artin L-series and prove basic properties about it before proceeding to define the Artin conductor and use it to prove a fundamental result concerning a functional equation for the analytic continuation of the L-series. We follow the treatment of [Neu] with regard to L-functions and the discussion in [Serb] of Artin conductors, while [FH, Sera] provide the proofs of the results we discuss in the theory of representations. In the following, all representations will be taken over $\mathbb{C}$, all extensions will be finite and assumed Galois unless explicitly stated otherwise. We always let $\mathcal{O}_K$ denote the ring of integers of K and in the local case we let k be the residue field.

2 Prerequisite Results

2.1 Elementary Representation Theory

In this section, we compile a few basic results about basic representation theory of finite groups over $\mathbb{C}$ that are necessary for the understanding of the following. Because the intent of the sequel is to introduce to Artin L-functions and not to representation theory, the results will be stated without proof. All proofs can be found either in the first section of [FH] or throughout [Sera]. For the remainder of this section, let G be a finite group, $H \subset G$ a subgroup, and V a finite dimensional vector space over $\mathbb{C}$. We begin with the definition of a representation.

Definition 1. A representation ρ of G is a homomorphism $\rho : G \to \mathrm{GL}(V)$. By abuse of notation, we will often refer to V as a representation of G, ommitting the explicit function. In this case, for $g \in G$, $v \in V$, we write $g \cdot v$ for $\rho(g)(v)$. A representation is irreducible if there is no $W \subset V$ a subspace that is preserved by the G action. The degree of a representation V is $\dim V$. With a representation V fixed, we define the character, $\chi_\rho : G \to \mathbb{C}^\times$ such that $\chi(g) = \mathrm{Tr}(\rho(g))$. We will often abbreviate χ_ρ to χ when context makes the representation obvious. A morphism of representations V to W is an element of $\mathrm{Hom}_G(V, W)$.

An easy example of a representation is the trivial representation that sends every element to the identity on V. In this case, for all $g \in G$, $\chi(g) = \dim V$. Note that for V of dimension greater than 1, this is never irreducible because any subspace is a subrepresentation. Another classic example is the regular representation, where $\dim V = |G|$ and V is thought of as the vector space on the elements of G where the G action is left multiplication. For G nontrivial, this representation is also never irreducible because the subspace $W = \{\sum_{g\in G} a_g v_g | \sum a_g = 0\} \subset V$ is clearly G-invariant. By Mashke's Theorem [FH, §1.2], all representations decompose into a finite direct sum of irreducible representations. We can say more about the regular representation. Let $r_G : V \to C^\times$ be the character of the regular representation. Then by the original fixed point formula of group actions, we see that the value $r_G(g)$ is just the number of basis elements fixed by a g action. Thus, $r_G(g) = 0$ if $g \neq 1$ and $g(1) = \dim V = |G|$. It is a basic fact of representation theory that the regular representation contains every irreducible representation W_i each with multiplicity $\dim W_i$. Thus if we let χ_1 denote the trivial character on G, we get that $r_G = \chi_1 + u_G$. We note that $u_G = \sum_{\chi \neq \chi_1} \chi(1)\chi$ is the character of a representation called the augmentation representation.

Recall from elementary linear algebra that Tr is constant on conjugacy class so for any $g, h \in G$, $\chi(g) = \chi(hgh^{-1})$. A function $\phi : G \to \mathbb{C}^\times$ that satisfies this property of being constant on conjugacy classes of G is called a class function. Consider the following, where ϕ, ψ are class functions on G

$$(\phi, \psi) = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}$$

Note that this is a Hermitian pairing on the space of class functions. This leads us to

Proposition 2. *Under the Hermitian pairing described above, the characters of irreducible representations form an orthonormal basis for the space of class functions. The number of irreducible representations is equal to the number of conjugacy classes in G. Moreover, a representation is determined by its character and an element in the space*

$$\{\sum a_r \chi_r | \chi_r \text{ are characters of irreducible representations}\}$$

corresponds to a representation if and only if each $a_r \in \mathbb{Z}_{\geq 0}$. Moreover, the multiplicity of the irreducible W in the representation V is given by (χ_V, χ_W)

Proof. See [FH, §2.2]. ■

Henceforth, we shall refer to a representation and its character interchangeably. An interesting application of the above is that the multiplicity of the trivial representation in V is just $\dim V^G$, the subspace consisting of all $v \in V$ such that $g \cdot v = v$ for all $g \in G$. Applying Proposition 2, we get

$$\dim V^G = (\chi_V, \chi_1) = \frac{1}{|G|} \sum_{g \in G} \chi(g)$$

There are a few basic operations on representations. Taking the direct sum of two representations V, V' gives a representation $V \oplus V'$ where $g \cdot (v \oplus v') = g \cdot v \oplus g \cdot v'$. Similarly with tensor products, exterior powers, dualizing, and symmetric powers. The characters behave nicely with respect to these operations:

Proposition 3. *Let V, V' be representations of G. Then*

$$\begin{aligned} \chi_{V \oplus V'} &= \chi_V + \chi_{V'} \\ \chi_{V \otimes V'} &= \chi_V \cdot \chi_{V'} \\ \chi_{V^\vee} &= \overline{\chi_V} \end{aligned}$$

Proof. See [FH, §2.1 Proposition 1] ■

Now we consider how representations interact with a change in group. Suppose we have a representation $\rho : G \to \mathrm{GL}(V)$ and a group homomorphism $h : H \to G$. If ϕ is a class function of G, then let $h^*(\phi) = \phi \circ h$ a class function on H. Then, we have the famed Frobenius Reciprocity Theorem:

Theorem 4 (Frobenius)**.** *In the situation above, for each class function ψ on H, there exists a unique class function on G, $h_*(\psi)$ that, for all class functions ϕ on G satisfies*

$$(\phi, h_*(\psi)) = (h^*(\phi), \psi)$$

Proof. See [Sera, §§7.1-7.3] or [FH, §4.3] for a special case of the above. ■

We apply Theorem 4 chiefly in two cases. First is when $H \subset G$ and the homomorphism is inclusion where ϕ is the character of a representation V of G and ψ is the character of a representation W of H. In this case, we set $h^*(\phi) = \phi|_H$ restriction and write it as such, or by abuse of notation, we sometimes just write ϕ for $h^*(\phi)$. Then, we have that $h_*(\psi)$ is just $\mathrm{Ind}_H^G(\psi)$ in the sense discussed in the context of group cohomology, which we write as ψ_*. Explicitly, we get

$$\psi_*(g) = \sum_{g'} \psi(g' g g'^{-1})$$

where g' runs over a set of representatives of left cosets of H and we extend by 0 outside of H.

The other case in which we use Theorem 4 is where G is a quotient of H and h is projection. In this case, we again abbreviate by writing $h^*(\phi) = \phi$ and $h_*(\psi) = \psi_*$ and trust to context to dispel confusion. In this case, if $G = H/N$, then we get explicitly

$$\psi_*(g) = \frac{1}{|N|} \sum_{g' \mapsto g} \psi(g')$$

Note that in the first case above, if ψ is the character of a representation of H, then ψ_* is the character of a representation of G, namely the induced representation. Similarly, in the second case, if ϕ is the character of a representation of G then $h^*(\phi)$ is the character of a representation of H, given by composing $\rho \circ h$. Finally, we state a beautiful result of Brauer's that will be relevant later:

Theorem 5 (Brauer). *Every character χ of G is a $\mathbb{Z}$-linear combination of characters induced from degree 1 characters associated to subgroups of G. In other words, there exist $a_1, ..., a_r \in \mathbb{Z}$ and $\chi_{1*}, ..., \chi_{r*}$ characters of G such that*

$$\chi = \sum_{i=1}^{r} a_i \chi_{i*}$$

and each χ_{i*} is induced from a class function χ_i on some subgroup $H_i \subset G$ such that χ_i is the character associated to a one-dimensional representation of H_i.

Proof. See [Bra] or [Sera, §10.5]. ∎

Representation theory does, of course, extend much beyond these few results, but the rest unfortunatley lies outside of the scope of the current paper.

2.2 Decomposition, Inertia, and Ramification Groups

We follow [Serb, §1.7] to introduce a few key concepts that will be required for the sequel. If L/K is a Galois extension of local fields and $G = \mathrm{Gal}(L/K)$ then for each $\mathfrak{p} \subset \mathcal{O}_K$, the group G acts transitively on the set of $\mathfrak{P} \subset \mathcal{O}_L$ lying over $\mathfrak{p}$. If $\mathfrak{P}|\mathfrak{p}$ then we define the decomposition group $D_{\mathfrak{P}} \subset G$ is the set of elements that preserve $\mathfrak{P}$. By the transitivity of the G action on the set of $\mathfrak{P}|\mathfrak{p}$ we have that if $\mathfrak{P}'|\mathfrak{p}$ then $D_{\mathfrak{P}}$ and $D_{\mathfrak{P}'}$ are conjugate. Thus, we can fix a $\mathfrak{P}|\mathfrak{p}$ and let $D = D_{\mathfrak{P}}$. Then, letting l and k denote the residue fields of $\mathcal{O}_L$ and $\mathfrak{O}_K$ respectively, then the decomposition group fixes $\mathfrak{P}$, so it acts naturally on $\mathcal{O}_L/\mathfrak{P} = l$. Thus we obtain a natural morhism $D \to \mathrm{Gal}(l/k)$ obtained simply by choosing a representative in $\mathcal{O}_L$ and taking the D action followed by reduction $\mod \mathfrak{P}$. We define the inertia subgroup, $I_{\mathfrak{P}}$ as the kernel of this morphism.

Recall that if L/K is a Galois extension of local fields and if $G = \mathrm{Gal}(L/K)$ then G acts on $\mathcal{O}_L$. Let x be an element such that $\mathcal{O}_L = \mathcal{O}_K[x]$. Then it is a result from [Serb, §IV.1] that the function $i_G(s) = v_L(s \cdot x - x)$ is independent of choice of x. We define the i^{th} ramification subgroup $G_i \subset G$ such that $G_i = \{g \in G | i_G(g) \geq i+1\}$. Then we have the following:

Proposition 6. *With G_i as defined above, the G_i form a decreasing sequence of normal subgroups of G and we have explicitly that $G_{-1} = G$ and G_0 is the inertia subgroup. Moreover, this sequence stabilizes for $i >> 0$ to the trivial group.*

Proof. See [Serb, §IV.1]. ∎

We extend our definition of ramification groups to the reals in such way that if $u \in \mathbb{R}_{>-2}$, then $G_u = G_i$, where $i \geq u > i - 1$ is an integer. Then we define the function

$$\phi_{L/K}(u) = \phi(u) = \int_0^u \frac{dt}{[G : G_t]}$$

We get elementary properties of ϕ:

Proposition 7. *With ϕ defined above, if $m \leq u < m+1$ then*

$$\phi(u) = \frac{|G_1| + |G_2| + ... + |G_m| + (u-m)|G_{m+1}|}{|G_0|}$$

Moreover, ϕ is continuous, piecewise linear, and increasing. If $L \supset K' \supset K$ a tower of fields, then $\phi_{L/K} = \phi_{K'/K} \circ \phi_{L/K'}$

Proof. See [Serb, §IV.3] ■

We have two more theorems that will be required:

Theorem 8 (Herbrand)**.** *If H is a normal subgroup of G, then for any u, we have*

$$G_u H/H = (G/H)_{\phi(u)}$$

Proof. See [Serb, §IV.3 Lemma 5] ■

Theorem 9 (Hasse-Arf)**.** *Let L/K be an abelian extension of a local field with $G = \mathrm{Gal}(L/K)$ such that the extension of residue fields l/k is seperable. Then for $i \in \mathbb{N}$ such that $G_i \neq G_{i+1}$ we have $\phi_{L/K}(i) \in \mathbb{Z}$.*

Proof. See [Serb, §V.7] ■

Now that we have collected enough elementary results, we proceed to the discussion of the Artin L-functions.

3 Definitions and Basic Properties

Recall that a Dirichlet L-series is defined for a character $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}$ and extended by 0 to give

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)p^{-s}}$$

This product expression provides the motivation for the development of the Artin L-series. First, let L/K be a Galois extension of number fields, $G = \mathrm{Gal}(L/K)$. Fix a representation (ρ, V) of G and a prime $\mathfrak{P}$ in L lying over $\mathfrak{p} \subset K$. Then by the discussion in Section 2.2 we can form the decomposition group $D_\mathfrak{P}$ and the inertia group $I_\mathfrak{P}$. Let $L_\mathfrak{P}/K_\mathfrak{p}$ be the extension of local fields associated to the primes $\mathfrak{P}|\mathfrak{p}$ and $l_\mathfrak{P}/k_\mathfrak{p}$ the extension of residue fields. By above, we have $D_\mathfrak{P}/I_\mathfrak{P} \cong \mathrm{Gal}(l_\mathfrak{P}/k_\mathfrak{p})$ canonically. This extension is a finite extension of finite fields so is cyclic and is canonically generated by $\phi_\mathfrak{P}$, the Frobenius automorphism that is the pull back of $(x \mapsto x^q) \in \mathrm{Gal}(l_\mathfrak{P}/k_\mathfrak{p})$ if $q = \mathrm{Nm}(\mathfrak{p})$. Just as in how in the definition of the inflation map we have to restrict to a representation invariant under the factor group, so too, do we have to do the same in defining a representation of $D_\mathfrak{P}/I_\mathfrak{P}$; thus, let $W_\mathfrak{P} = V^{I_\mathfrak{P}}$ be the subrepresentation of V that is invariant under action by $I_\mathfrak{P} \subset G$ and consider the representation induced by ρ on $W_\mathfrak{P}$. Thus we get $\overline{\rho}(\phi_\mathfrak{P}) \in \mathrm{GL}(W_\mathfrak{P})$ and we can take the characteristic polynomial $P_\mathfrak{P}(t) = \det(1 - t\overline{\rho}(\phi_\mathfrak{P}))$. By the discussion in Section 2.2, if we were to choose another $\mathfrak{P}'$ lying over $\mathfrak{p}$ then because the decomposition and inertia groups are conjugate and characteristic polynomials of similar matrices are equal, we get the same $P_\mathfrak{P}(t)$ so the characteristic polynomial is independent of our choice in prime lying over $\mathfrak{p}$. We can now finally define an Artin L-series

Definition 10. Let L/K be a Galois extension of number fields and ρ a representation of $G = \text{Gal}(L/K)$. Then we define an Artin L-series as

$$\mathcal{L}(L/K, \rho, s) = \prod_{\mathfrak{p}} \frac{1}{P_{\mathfrak{P}}(\text{Nm}(\mathfrak{p})^{-s})}$$

Where the product is over all finite places of K and $\mathfrak{P}|\mathfrak{p}$ is a prime in L lying over $\mathfrak{p}$.

Remark 11. Note that by the discussion above, the choice of $\mathfrak{P}|\mathfrak{p}$ does not matter and Definition 10 is well defined, assuming convergence. Also, as noted in Proposition 2, because a representation is uniquely determined by its character, we will associate the character χ to the representation ρ and use $\mathfrak{L}(L/K, \chi, s)$ to mean $\mathfrak{L}(L/K, \rho, s)$ and use these interchangeably.

Before we make a few general remarks, let us consider how this is a generalization of the Dirichlet L-functions. Suppose we have a character $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}$; we can form the series

$$\mathcal{L}(\chi, s) = \prod_{p} \frac{1}{1 - \chi(p)p^{-s}}$$

Now, let $K = \mathbb{Q}$ and $L = \mathbb{Q}(\zeta_m)$ a cyclotomic extension. Then, either by elementary algebra or a very easy application of class field theory, we get an isomorphism $G = \text{Gal}(L/K) \cong (\mathbb{Z}/m\mathbb{Z})^\times$ whose inverse is $a \mapsto (a \mapsto \zeta_m^a)$. Thus we can consider χ to be a character on G and so we can form the Artin L-series above $\mathcal{L}(L/K, \chi, s)$. This character gives us a one dimensional representation and we verify immetiately that $P_{\mathfrak{P}}(t) = 1 - \chi(p)t$. Thus, in the case of cyclotomic extensions, we recover the Dirichlet L-series from the Artin L-series.

More generally, while the definition above, as noted in the remark, does not depend on choice of prime lying over each $\mathfrak{p}$, there is still an issue of convergence.

Proposition 12. *For any L/K Galois extension of number fields and any representation ρ (or character χ), the function $\mathcal{L}(L/K, \chi, s)$ converges absolutely and unifromly for* $\text{Re}(s) \geq 1 + \epsilon$ *for all $\epsilon > 0$. Thus $\mathcal{L}(L/K, \chi, s)$ defines a holomorphic function in the half plane* $\text{Re}(s) > 1$

Proof. The proof of this is somewhat long and analytic in flavor without the redeeming feature of being very relevant to the sequel. As such, we will cite it as a combination of [Neu, §VII.8 Proposition 1] and remarks made in [Neu, p. 518]. ■

Remark 13. When χ is the trivial character, we recover the Dedekind ζ function for the number field K. To see this, note that for the trivial representation, for each $\mathfrak{P}$, $P_{\mathfrak{P}}(t) = 1 - t$. Thus, by definition, we have

$$\mathcal{L}(L/K, \chi, s) = \prod_{\mathfrak{p}} \frac{1}{1 - \text{Nm}(\mathfrak{p})^{-s}} = \zeta_K(s)$$

For the special case, L a cyclotomic extension of $K = \mathbb{Q}$ this result is encapsulated in the discussion above relating Aritn L-functions to Dirichlet L-functions.

Now that we have seen some basic examples of L functions and seen that they converge, the natural next question is how they interact with changes in input. We get the following proposition:

Proposition 14. *Let L/K be a Galois extension of number fields, $G = \text{Gal}(L/K)$. Then:*

1. *If χ_1, χ_2 are two characters of G, then*

$$\mathcal{L}(L/K, \chi_1 + \chi_2, s) = \mathcal{L}(L/K, \chi_1, s)\mathcal{L}(L/K, \chi_2, s)$$

2. *If L'/K is Galois and $L' \supset L \supset K$ and χ is a character of G then*

$$\mathcal{L}(L'/K, \chi, s) = \mathcal{L}(L/K, \chi, s)$$

where χ is a character on $\text{Gal}(L'/K)$ *by Theorem 4.*

3. *If K' a subextension $L \supset K' \supset K$ and χ is a character on* $\mathrm{Gal}(K'/K)$ *then by Theorem 4 we have a character χ_* on G and*
$$\mathcal{L}(L/K, \chi_*, s) = \mathcal{L}(K'/K, \chi, s)$$

Proof. 1): Let V, V' be representations of G with characters χ, χ'. Then $V \oplus V'$ is a representation of G with character $\chi + \chi'$ by Proposition 3. Moreover, $\det(1 - (\rho \oplus \rho')(\phi_{\mathfrak{P}})t) = \det(1 - \rho(\phi_{\mathfrak{P}})t)\det(1 - \rho'(\phi_{\mathfrak{P}})t)$ so the result follows.

2): Let $\mathfrak{P}'|\mathfrak{P}|\mathfrak{p}$ be a primes of L', L, and K respectively, each lying over the next. Let χ be the character associated to the representation V of $G = \mathrm{Gal}(L/K)$. By Application 2 of Theorem 4, we have that $\mathrm{Gal}(L'/K)$ acts on V because $\mathrm{Gal}(L'/K) \to G$ is projection. This projection induces

$$\begin{aligned} G_{\mathfrak{P}'} &\to G_{\mathfrak{P}} \\ I_{\mathfrak{P}'} &\to I_{\mathfrak{P}} \\ G_{\mathfrak{P}'}/I_{\mathfrak{P}'} &\to G_{\mathfrak{P}}/I_{\mathfrak{P}} \end{aligned}$$

We notice immediately that the last sends $\phi_{\mathfrak{P}'} \mapsto \phi_{\mathfrak{P}}$, but by Theorem 4, we then have an equivalence of representations and we get that $P_{\mathfrak{P}'}(t) = P_{\mathfrak{P}}(t)$. The result follows immediately.

3): This is similar to the proof of 2 and 1 in that one uses the representation associated to χ and the functoriality properties of Theorem 4 to derive the result on $P_{\mathfrak{P}}(t)$ and then conclude from there. Because it is similar to the above and somewhat long, we direct the interested reader to [Neu, §VII.10 Proposition 4]. ∎

Remark 15. Applying Proposition 14 to the case L cyclotomic over $K = \mathbb{Q}$ we recover the result on Dirichlet L-functions on sums of characters. Moreover, if $L' \supset L \supset K$ is a tower of cyclotomic extensions and $L = \mathbb{Q}(\zeta_m)$ then $L' = \mathbb{Q}(\zeta_n)$ where $m|n$ and we get that if χ' is a character of $(\mathbb{Z}/n\mathbb{Z})^\times$ corresponding to χ, then $\mathcal{L}(\chi, s) = \mathcal{L}(\chi', s)$. This is, of course, trivial considering the sum expression for $\mathcal{L}(\chi, s)$ because reduction $\pmod m$ is the same as reduction first $\pmod n$ and then $\pmod m$. The analogous result is clear for Proposition 14.3.

We have now developed enough machinery to apply the above to a nice result.

Corollary 16. *For L/K a Galois extension of number fields, we have the relation between the Dedekind ζ-functions*
$$\zeta_L(s) = \zeta_K(s) \prod_{\chi \neq 1} \mathcal{L}(L/K, \chi, s)^{\chi(1)}$$
where the χ run over the nontrivial irreducible characters of $G = \mathrm{Gal}(L/K)$.

Proof. By Theorem 4, we have that if χ_1 is the trivial character on $\{1\} \subset G$, then $\chi_{1_*} = r_G$, the character of the regular representation. As mentioned in Section 2.1, $r_G = \sum_\chi \chi(1) \cdot \chi$. We apply Proposition 14.3 to get $\mathcal{L}(L/L, \chi_1, s) = \mathcal{L}(L/K, \chi_{1_*}, s)$. But by Remark 13, $\mathcal{L}(L/L, \chi_1, s) = \zeta_L(s)$ and by Proposition 14.1, we have

$$\begin{aligned} \mathcal{L}(L/K, \chi_{1_*}, s) = \mathcal{L}(L/K, r_G, s) &= \mathcal{L}(L/K, \sum_\chi \chi(1) \cdot 1, s) \\ = \prod_\chi \mathcal{L}(L/K, \chi, s)^{\chi(1)} = \mathcal{L}(L/K, \chi_1, s) \cdot &\prod_{\chi \neq 1} \mathcal{L}(L/K, \chi, s)^{\chi(1)} \\ &= \zeta_K(s) \prod_{\chi \neq 1} \mathcal{L}(L/K, \chi, s)^{\chi(1)} \end{aligned}$$
∎

This seemingly simple result is the key to understanding a conjecture of Artin. The conjecture itself is

Conjecture 1. If χ is a nontrivial irreducible character of $G = \mathrm{Gal}(L/K)$ for L/K a Galois extension of number fields, then $\mathcal{L}(L/K, \chi, s)$ can be analytically continued to an entire function on $\mathbb{C}$

We will later show that this holds for abelian extensions. In point of fact, as [Neu] notes, Proposition 16 has important historical significance. Artin began his study of L-series precisely because he wished to know if, for any Galois extension of number fields L/K, the quotient $\zeta_L(s)/\zeta_K(s)$ is an entire function. The corollary relates this to Conjecture 1, and so, when we show below that Conjecture 1 holds for abelian extensions, we will have shown that Artin's original question holds for the same. Before we do this, we must first digress and discuss the Conductor.

4 Conductors

The idea behind and applications of Artin Conductors are subtle. We first treat the local case following the treatment of [Serb, §VI.2] and then pass to the global case.

4.1 Local Case

In the following, let L/K be a Galois extension of local fields, $G = \mathrm{Gal}(L/K)$ and let f be the inertial degree of L/K, i.e., $f = [l : k]$, the degree of the extension of residue fields. We begin by defining a function, $a_G : G \to \mathbb{C}^\times$ in terms of the function i_G discussed above in Section 2.2. Let

$$a_G(g) = \begin{cases} -f i_G(g) & g \neq 1 \\ f \sum_{h\neq 1} i_G(h) & g = 1 \end{cases}$$

By remarks in Section 2.2, we note that $a_G(s) = a_g(gsg^{-1})$ for all $g, s \in G$ and from the definition it is clear that $a_G(g) = a_G(g^{-1})$. We have that a_G is a class function on G and by construction we have that if χ_1 is the trivial character, then $(a_G, \chi_1) = 0$. Thus if we let X be the set of irreducible characters of G, we get

$$a_G = \sum_{\chi \in X} c_\chi \cdot \chi$$

for $c_\chi \in \mathbb{C}$ by remarks from Section 2.1. Moreover, for $\chi \in X$, we get

$$c_\chi = (a_G, \chi) = \frac{1}{|G|} \sum_{g\in G} a_G(g)\overline{\chi(g)}$$

$$= \frac{1}{|G|} \sum_{g\in G} a_G(g)\chi(g^{-1}) = \frac{1}{|G|} \sum_{g\in G} a_G(g^{-1})\chi(g) = \frac{1}{|G|} \sum_{g\in G} a_G(g)\chi(g)$$

Thus $c_\chi = (\chi, a_G)$. Let us define for any class function ϕ on G, $f(\phi) = (\phi, a_G)$. Then we have $a_G = \sum f(\chi)\chi$. We wish to show that $f(\chi)$ is a nonnegative integer for all $\chi \in X$. As a side note, we remark that this will give us that a_G is the character of a representation of G, called the Artin Representation. Before doing this, we need to develop a little bit of theory. First, let us define for a class function ϕ,

$$\phi(G_i) = \frac{1}{|G_i|} \sum_{g\in G_i} \phi(g)$$

Now, we have a computation relating the above:

Proposition 17. *Let G_i be as in Section 2.2 and let u_i be the character of the augmentation representation of G_i, as in Section 2.1. then u_{i_*} is a character on G and we have*

$$a_G = \sum_{i=0}^{\infty} \frac{1}{[G_0 : G_i]} u_{i_*}$$

Moreover, for ϕ a class function on G,

$$f(\phi) = \sum_{i=0}^{\infty} \frac{|G_i|}{|G_0|}(\phi(1) - \phi(G_i))$$

Proof. We know that G_i is normal so if $g \notin G_i$ then $u_{i_*}(g) = 0$ while if $g \in G_i$ and $g \neq 1$ then we get by definition of the augmentation character that $u_{i_*}(g) = -\frac{|G|}{|G_i|} = -f\frac{|G_0|}{|G_i|}$ and the orthogonality relation gives $\sum_{g \in G} u_{i_*}(g) = 0$. Thus, for $g \in G_i - G_{i+1}$ we have that the right hand side above is just $-f(i+1)$, which is exactly $a_G(g)$ by definition. Orthogonality relations on a_G and u_{i_*} yield the case $g = 1$. Note that $\phi(G_i)$ is constructed in such a way as to make $(\phi_{G_i}, u_i) = \phi(1) - \phi(G_i)$. By Theorem 4 we then get that $(\phi, u_{i_*}) = \phi(1) - \phi(G_i)$. Unwinding definitions gives us the desired result. ■

The above gives us the immediate application by substituting χ in for ϕ. Then we have

$$\begin{aligned} \chi(G_i) &= \frac{1}{|G_i|} \sum_{g \in G_i} \chi(g_i) \\ &= (\chi, \chi_1)_{G_i} = \dim V^{G_i} \end{aligned}$$

where the last equality comes from a remark made in Section 2.1. But $\chi(1) = \dim V$ so $\chi(1) - \chi(G_i) = \operatorname{codim} V^{G_i}$. Plugging in, we get

$$f(\chi) = \sum_i \frac{|G_i|}{|G_0|} \operatorname{codim} V^{G_i} \tag{1}$$

The above when put together yields:

Proposition 18. *If χ is a character of G then $f(\chi)$ is nonnegative and rational.*

Proof. By above we have that $|G_0| f(\chi)$ is the sum of nonnegative integers so is a nonnegative integer if χ is irreducible. Characters of representations are nonnegative integral linear combinations of irreducible characters so the result follows. ■

We have now developed enough theory to show that $f(\chi)$ is a nonnegative integer.

Theorem 19. *As defined above, $f(\chi)$ is a nonnegative integer. Thus, a_G is the character of a representation of G.*

Proof. We first show the result for a character χ of degree 1. Let c_χ be the largest integer such that $\chi_{G_{c_\chi}}$ is nontrivial. Then we first show that $f(\chi) = \phi_{L/K}(c_\chi) + 1$. This is easy to see because if $i \leq c_\chi$ then $\chi(G_i) = 0$ by orthogonality relations so $\chi(1) - \chi(G_i) = 1$ and similarly if $i > c_\chi$ then $\chi(G_i) = 1$ and so $\chi(1) - \chi(G_i) = 0$ because χ is a degree 1 character. Now applying Proposition 17, we get

$$\begin{aligned} f(\chi) &= \sum_{i=0}^{\infty} \frac{|G_i|}{|G_0|} (\phi(1) - \phi(G_i)) \\ &= \sum_{i=0}^{c_\chi} \frac{|G_i|}{|G_0|} = \phi_{L/K}(c_\chi) + 1 \end{aligned}$$

where the last equality comes from Proposition 7. Now, if $H = \ker \chi$, let K'/K such that $K' = L^H$ and let d_χ the largest integer such that $(G/H)_{d_\chi} \neq 1$. By Theorem 8 we have that $d_\chi = \phi_{L/K'}(c_\chi)$ and by Proposition 7 we have $\phi_{L/K}(c_\chi) = \phi_{K'/K} \circ \phi_{L/K'}(c_\chi) = \phi_{K'/K}(d_\chi)$. Thus we have that $f(\chi) = \phi_{K'/K}(d_\chi) + 1$. Note that $G/H \subset \mathbb{C}^\times$ so we can apply Theorem 9 to see that $\phi_{K'/K}(d_\chi) \in \mathbb{Z}$. This is nonnegative by Proposition 18.

Now we generalize to higher degree characters of G. We know by Proposition 18 that $f(\chi)$ is a nonnegative rational. We now apply Theorem 5 to get that there exist subgroups $H_i \subset G$ with degree 1 characters χ_i such that $\chi = \sum a_i \chi_{i_*}$. By above, we already have that $f(\chi_{i_*})$ is integral so χ is a $\mathbb{Z}$-linear combination of characters that evaluate to integers under f. By Proposition 18 it is nonnegative so we are done. ■

We can now define the local Artin conductor

Definition 20. If L/K is a Galois extension of local fields, then we define the local Artin Conductor of the character χ of $G = \operatorname{Gal}(L/K)$

$$\mathfrak{f}_{\mathfrak{p}}(\chi) = \mathfrak{p}^{f(\chi)}$$

where $\mathfrak{p}$ is the maximal ideal in $\mathcal{O}_K$

In the general case, the conductor is defined to be $\mathfrak{p}^n$, where n is the smallest integer such that G_n is trivial. The reason that the Artin conductor is also called the conductor is that, as proven in [Neu, §VII.11 Proposition 6], if χ is a character of $G = \mathrm{Gal}(L/K)$ and L_χ is the fixed field of $\ker \chi$, then $\mathfrak{f}_\mathfrak{p}(\chi)$ is the conductor in the usual sense of L_χ/K. This also follows almost immediately from the above discussion. We now proceed to the global case.

4.2 Global Case

We now work in the global case, so let L/K be a Galois extension of number fields, and $\mathfrak{p}$ a prime of K, with $\mathfrak{P}$ a prime of L lying over $\mathfrak{p}$. Then, locally, we have the completion $L_\mathfrak{P}/K_\mathfrak{p}$ and we denote by $G_\mathfrak{P} \subset \mathrm{Gal}(L_\mathfrak{P}/\mathfrak{p})$ the decomposition group of this local extension associated with $\mathfrak{P}$. Above we defined the function a_G for a local extension, so let $a_\mathfrak{P} = a_{G_\mathfrak{P}}$ be the function on $G_\mathfrak{P}$ extended to $G = \mathrm{Gal}(L/K)$ by setting it equal to 0 outside of $G_\mathfrak{P}$. Then we can define a class function

$$a_\mathfrak{p} = \sum_{\mathfrak{P}|\mathfrak{p}} a_\mathfrak{P}$$

which, by Theorem 4, is just $a_{\mathfrak{P}_*}$. By the discussion after Theorem 4, because $a_\mathfrak{P}$ is the character of a representation of $G_\mathfrak{P}$ by Theorem 19, we have that $a_\mathfrak{p}$ is the character of a representation of G. In the local case, the definition of f was easy because there was only one prime to keep track of. Now, the situation is a little bit more complicated and we define

$$f(\chi, \mathfrak{p}) = (\chi, a_\mathfrak{p})$$

From the concluding remarks of the previous section and the relationship between ramification and the decomposition group, we see that if $\mathfrak{p}$ is unramified that we must have $f(\chi, \mathfrak{p}) = 0$. We can now generalize our local conductor to $\mathfrak{f}_\mathfrak{p}(\chi) = \mathfrak{p}^{f(\chi,\mathfrak{p})}$. By the preceding comment, if $\mathfrak{p}$ is unramified we have $\mathfrak{f}_\mathfrak{p}(\chi) = 1$. We can now generalize Definition 20 to the global case:

Definition 21. For L/K a Galois extension of number fields, we define the global Artin conductor associated to a character of $G = \mathrm{Gal}(L/K)$, χ, to be

$$\mathfrak{f}(L/K, \chi) = \mathfrak{f}(\chi) = \prod_{\mathfrak{p} \nmid \infty} \mathfrak{f}_\mathfrak{p}(\chi)$$

where the product runs over all finite primes of K. Note that because there are only a finite number of ramified primes, the product above is defined.

Remark 22. There is a famous result relating the conductor to an ideal called the discriminant, proven in [Serb, §VI.3 Corollary 2] and [Neu, §VII.11 Theorem 9] that relates the discriminant, $\mathfrak{d}_{L/K}$ of an extension to the conductor. In a somewhat vacuous way, since we have not taken the time to develop a theory of discriminants, we will simply define the discriminant as

$$\mathfrak{d}_{L/K} = \prod \mathfrak{f}(\chi)$$

where the product ranges over all irreducible characters of $G = \mathrm{Gal}(L/K)$, which is finite by remarks from Section 2.1.

We conclude our discussion of conductors by relating a proposition similar in form to Proposition 14 but for conductors.

Proposition 23. *Let L/K be a Galois extension of number fields with $G = \mathrm{Gal}(L/K)$. Then*

1. *For characters χ, χ' of G, we have $\mathfrak{f}(\chi + \chi') = \mathfrak{f}(\chi)\mathfrak{f}(\chi')$.*
2. *If χ_1 is the trivial character, we have $\mathfrak{f}(\chi_1) = (1)$, the unit ideal.*

3. *If $L' \supset L \supset K$ is Galois over K and χ is a character of* $\mathrm{Gal}(L'/K)$*, then*

$$\mathfrak{f}(L'/K, \chi) = \mathfrak{f}(L/K, \chi)$$

4. *If $H \subset G$ is a subgroup and $K' = L^H$ and χ is a character of H then χ_* is a character of G and*

$$\mathfrak{f}(L/K, \chi_*) = \mathfrak{d}_{K'/K}^{\chi(1)} \mathrm{Nm}_{K'/K}(\mathfrak{f}(L/K', \chi))$$

Proof. The first three are easy, simply applying the definitions. The proof of the fourth is long and not fully relevant to the following, so we cite [Neu, §VII.11 Proposition 7] or [Serb, §VI.3 Proposition 6]. ■

Finally, we define a number associated to the character, $c(L/K, \chi)$ by noting that there exists an ideal in $\mathbb{Z}$:

$$\mathfrak{c}(L/K, \chi) = \mathfrak{d}_{K/\mathbb{Q}}^{\chi(1)} \mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{f}(L/K, \chi))$$

and setting $c(L/K, \chi)$ to be the positive generator of this ideal, giving us

$$c(L/K, \chi) = |d_K|^{\chi(1)} |\mathcal{O}_K / \mathfrak{f}(L/K, \chi)|$$

where d_K is the numberical discriminant of the number field K. An immediate application of Proposition 23 is the analogue for $c(L/K, \chi)$ of Propositions 14 and 23

Proposition 24. *Let L/K a Galois extension of number fields with Galois group $G = \mathrm{Gal}(L/K)$. Then*

1. *If χ, χ' are characters of G then $c(L/K, \chi + \chi') = c(L/K, \chi)c(L/K, \chi')$.*

2. *If χ_1 is the trivial character then $c(L/K, \chi_1) = |d_K|$*

3. *If $L' \supset L \supset K$ is Galois then*
$$c(L'/K, \chi) = c(L/K, \chi)$$

4. *If $L \supset K' \supset K$ is Galois and χ a character of G then*

$$c(L/K, \chi_*) = c(L/K', \chi)$$

Now that we have introduced the Artin conductor and how it transforms with change in field and character, we are ready to apply the work above to the desired results pertaining to Conjecture 1.

5 The Functional Equation and Beyond

5.1 Artin's Conjecture and Abelian Extensions

Recall that in the course of attempting to prove a result regarding quotients of Dedekind ζ functions, Artin conjectured

Conjecture 1. If χ is a nontrivial irreducible character of $G = \mathrm{Gal}(L/K)$ for L/K a Galois extension of number fields, then $\mathcal{L}(L/K, \chi, s)$ can be analytically continued to an entire function on $\mathbb{C}$

We prove this for abelian extensions. For the remainder of the section, let L/K be an abelian extension of global field with $G = \mathrm{Gal}(L/K)$. We first need to introduce the Artin symbol, a map

$$\left(\frac{L/K}{\cdot}\right) : J_K^{\mathfrak{m}} \to \mathrm{Gal}(L/K)$$

for some modulus $\mathfrak{m}$. First, note that the decomposition group $\mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}}) \subset G$ is generated by the Frobenius automorphism $\phi_{\mathfrak{p}}$ by class field theory. Thus, for primes $\mathfrak{p} \in J_K^{\mathfrak{m}}$, ideals in K prime to the modulus $\mathfrak{m}$, let

$$\left(\frac{L/K}{\mathfrak{p}}\right) = \phi_p$$

More generally, for any ideal $\mathfrak{a} \in J_K^{\mathfrak{m}}$, we can express

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{n_\mathfrak{p}}$$

where all but finitley many $n_\mathfrak{p}$ are zero. Then we define

$$\left(\frac{L/K}{\mathfrak{a}}\right) = \prod_{\mathfrak{p}} \left(\frac{L/K}{\mathfrak{p}}\right)^{n_\mathfrak{p}}$$

Note that this is constructed as a homomorphism $J_K^{\mathfrak{m}} \to G$. The fundamental result is

Theorem 25. *Let L/K be abelian and suppose $\mathfrak{m}$ is a modulus such that $L \subset L'$, where L' is the ray class field for $\mathfrak{m}$. Then the Artin symbol*

$$\left(\frac{L/K}{\cdot}\right) : J_K^{\mathfrak{m}} \to \mathrm{Gal}(L/K)$$

is a surjective homomorphism whose kernel includes the group of principal ideals prime to $\mathfrak{m}$, $P_K^{\mathfrak{m}}$.

Proof. See [Neu, §VI.7 Theorem 1]. ∎

Now, let $\mathfrak{f} = \mathfrak{f}(\chi_1)$ be the conductor of the extension L/K, where χ_1 is the trivial character on G. Then L lies in the ray class field of $\mathfrak{f}$ by results from [Neu, §VI.6] and so, by Theorem 25, we have a surjective homomorphism

$$J^{\mathfrak{f}}/P^{\mathfrak{f}} \to G$$
$$\mathfrak{a} \mapsto \left(\frac{L/K}{(\mathfrak{a})}\right)$$

where $J^{\mathfrak{f}}$ is the group of ideals in K prime to $\mathfrak{f}$ and $P^{\mathfrak{f}}$ is the subgroup of $J^{\mathfrak{f}}$ of principal ideals prime to $\mathfrak{f}$. If χ is an irreducible character of G, then $\chi \circ \left(\frac{L/K}{\cdot}\right)$ is a character of $Cl_{\mathfrak{f}} = J^{\mathfrak{f}}/P^{\mathfrak{f}}$. By Theorem 4, we have an induced $\tilde{\chi}$ a degree 1 character of $J^{\mathfrak{f}}$. We also cite [Neu, §VII.8 Theorem 5] that there exists a *Hecke L-series*, $L(\tilde{\chi}, s)$ associated with $\tilde{\chi}$ that is an entire function on $\mathbb{C}$. Using all of the above, we note

Theorem 26. *If L/K is an abelian extension, $G = \mathrm{Gal}(L/K)$, $\mathfrak{f}$ the conductor, then for each nontrivial irreducible character χ and associated $\tilde{\chi}$ as above, then leting $S_\chi = \{\mathfrak{p}|\mathfrak{f} : \chi(I_\mathfrak{P}) = 1\}$ we get*

$$\mathcal{L}(L/K, \chi, s) = \prod_{\mathfrak{p} \in S_\chi} \frac{1}{P_\mathfrak{P}(\mathrm{Nm}(\mathfrak{p})^{-s})} L(\tilde{\chi}, s)$$

Proof. See [Neu, §VII.10 Theorem 6]. ∎

Using the above result, we prove Conjecture 1 for the abelian case:

Theorem 27. *If L/K is an abelian extension of number fields with $G = \mathrm{Gal}(L/K)$ then for all nontrivial characters χ of G the function $\mathcal{L}(L/K, \chi, s)$ can be analytically continued to an entire function.*

Proof. If χ is injective, then by Theorem 26, $S_\chi = \emptyset$ and we get the equality $\mathcal{L}(L/K, \chi, s) = L(\tilde{\chi}, s)$, which is entire by [Neu, §VII.8 Theorem 5]. But if χ is a character of G, then there exists L_χ, the fixed field of the kernel of χ and $\chi : L_\chi \to \mathbb{C}^\times$ is injective so $\mathcal{L}(L_\chi/K, \chi, s)$ is entire. By Proposition 14, we have $\mathcal{L}(L/K, \chi, s) = \mathcal{L}(L_\chi/K, \chi, s)$ so we are done. ∎

Thus we have a fundamental application of class field theory to Artin's conjecture, showing that he is correct in the case of abelian extensions. We now proceed to discuss the functional equation.

5.2 Functional Equation

We wish to establish a functional equation for Artin L-functions that opens up the exploration of this class of functions to an plethora of new results. First, though, we need to 'complete' the Artin L-function. In some sense, the function to date only expresses data from the finite places and completing it allows us to have a function that informs us as to what happens at the infinite places. To express this, note that we have defined

$$\mathcal{L}(L/K, \chi, s) = \prod_{\mathfrak{p} \nmid \infty} \frac{1}{P_{\mathfrak{P}}(\mathrm{Nm}(\mathfrak{p}^{-s}))}$$

In a sense, we can think of each factor as a function $L_{\mathfrak{p}}(L/K, \chi, s)$; we want a similar factor for the infinite primes. In order to do this, let

$$L_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$$
$$L_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$$

where $\Gamma(s)$ is the Γ function studied in any introduction to complex analysis course. Let $\phi_{\mathfrak{P}}$ be the distinguished generator of $\mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}})$ and let

$$n^+ = \frac{\chi(1) + \chi(\phi_{\mathfrak{P}})}{2}$$
$$n^- = \frac{\chi(1) - \chi(\phi_{\mathfrak{P}})}{2}$$

Then for $\mathfrak{p}$ and infinite prime, let

$$\mathcal{L}_{\mathfrak{p}}(L/K, \chi, s) = \begin{cases} L_{\mathbb{C}}(s)^{\chi(1)} & \mathfrak{p} \text{ complex} \\ L_{\mathbb{R}}(s)^{n^+} L_{\mathbb{R}}(s+1)^{n^-} & \mathfrak{p} \text{ real} \end{cases}$$

Remark 28. Despite the fact that the above definition seems rather unintuitive, the intuition comes from the representation theory. In fact, $\phi_{\mathfrak{P}}$ is an involution on the representation V associated to χ and has two eigenspaces, V^+ associated to an eigenvalue of 1 and V^- associated to an eigenvalue of -1. An elementary computation will show that $\dim V^+ = n^+$ and $\dim V^- = n^-$. The rest is constructed to satisfy the following functoriality properties.

As with regular Artin L-functions we had Proposition 14 and with the conductor and the $c(L/K, \chi)$ we had Propositions 23 and 24 repectively, we have functoriality properties of the $\mathcal{L}_{\mathfrak{p}}$ as well:

Proposition 29. *For L/K a Galois extension with $G = \mathrm{Gal}(L/K)$ and $\mathfrak{p}$ an infinite prime, we have*

1. *If χ, χ' are characters of G then $\mathcal{L}_{\mathfrak{p}}(L/K, \chi + \chi', s) = \mathcal{L}_{\mathfrak{p}}(L/K, \chi, s)\mathcal{L}_{\mathfrak{p}}(L/K, \chi', s)$.*
2. *If $L' \supset L \supset K$ is Galois then*
$$\mathcal{L}_{\mathfrak{p}}(L/K, \chi, s) = \mathcal{L}_{\mathfrak{p}}(L'/K, \chi, s)$$
.
3. *If $L \supset K' \supset K$ is Galois and χ is a character of $\mathrm{Gal}(L/K')$ then*
$$\mathcal{L}_{\mathfrak{p}}(L/K, \chi_*, s) = \prod_{\mathfrak{q} \mid \mathfrak{p}} \mathcal{L}_{\mathfrak{q}}(L/K', \chi, s)$$
where $\mathfrak{q}$ runs over all primes of K' lying over $\mathfrak{p}$.

Proof. In the interest of brevity, we will not prove this functoriality claim again. See [Neu, §VII.12 Proposition 1] for details. ∎

We can now define

$$\mathcal{L}_\infty(L/K,\chi,s) = \prod_{\mathfrak{p}|\infty} \mathcal{L}_\mathfrak{p}(L/K,\chi,s)$$

and clearly $\mathcal{L}_\infty(L/K,\chi,s)$ obeys the same functoriality properties in Proposition 29. We can now, finally, complete the Artin L-function:

Definition 30. The completed Artin L-function of the character χ of G is

$$\Lambda(L/K,\chi,s) = c(L/K,\chi)^{\frac{s}{2}}\mathcal{L}_\infty(L/K,\chi,s)\mathcal{L}(L/K,\chi,s)$$

It is clear from the definition that, because the right hand side obeys all of the functoriality properties present in Propositions 14, 23, 24 and 29, we have the $\Lambda(L/K,\chi,s)$ obeys these as well. Now, we define the completed Hecke series associated to the character $\tilde{\chi}$ associated to χ as

$$\Lambda(\tilde{\chi},s) = (|d_K|\operatorname{Nm}(\mathfrak{f}(\tilde{\chi})))^{\frac{s}{2}}|L_\infty(\tilde{\chi},s)L(\tilde{\chi},s)$$

A discussion of Hecke series lies outside the scope of this paper, but the interested reader is referred to [Neu, §VII.8]. The part that is relevant to us is

Proposition 31. *Let L/K be Galois with $G = \operatorname{Gal}(L/K)$ and χ a character of G and $\tilde{\chi}$ the associated character as in Theorem 26. Then $\Lambda(L/K,\chi,s) = \Lambda(\tilde{\chi},s)$. Moreover, $\Lambda(\tilde{\chi},s)$ satisfies the functional equation*

$$\Lambda(\tilde{\chi},s) = W(\tilde{\chi})\Lambda(\overline{\tilde{\chi}},1-s)$$

where $W(\tilde{\chi})$ is a constant in $\mathbb{C}^\times$ of norm 1.

Proof. For the first result see [Neu, §VII.12 Proposition 5] and for the second see [Neu, §VII.8 Theorem 6]. ■

We are now prepared to state and prove the desired functional equation:

Theorem 32. *For L/K Galois extension of number fields and χ a character of $G = \operatorname{Gal}(L/K)$, $\Lambda(L/K,\chi,s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies:*

$$\Lambda(L/K,\chi,s) = W(\chi)\Lambda(L/K,\overline{\chi},1-s)$$

where $W(\chi)$ is a constant in $\mathbb{C}$ of norm 1.

Proof. We apply Theorem 5 to note that there exist subgroups $H_i = \operatorname{Gal}(L/K_i)$ and degree 1 characters of H_i, χ_i such that there are integers n_i and $\chi = \sum n_i\chi_{i_*}$. From Proposition 31 and functoriality, we have

$$\Lambda(L/K,\chi,s) = \prod_i \Lambda(L/K,\chi_{i_*},s)^{n_i} = \prod_i \Lambda(L/K_i,\chi_i,s)^{n_i} = \prod_i \Lambda(\tilde{\chi_i},s)^{n_i}$$

By Proposition 31, we have

$$\Lambda(\tilde{\chi_i},s) = W(\tilde{\chi_o})\Lambda(\overline{\tilde{\chi_i}},1-s)$$

So, if we let

$$W(\chi) = \prod_i W(\tilde{\chi_i})$$

we still have $|W(\chi_i)| = 1$ and we have, again by Proposition 31,

$$\Lambda(\overline{\tilde{\chi_i}},1-s) = \Lambda(L/K,\overline{\chi_i},1-s)$$

and so we get

$$\Lambda(L/K,\chi,s) = W(\chi)\prod_i \Lambda(L/K,\overline{\chi_i},1-s) = W(\chi)\Lambda(L/K,\overline{\chi},1-s)$$

where the last equality is given by functoriality. ■

Remark 33. Note that the reason that the proof above worked was because each factor of $\Lambda(L/K, \chi, s)$ changes with respect to character and field extension in the identical manner. As such, our somewhat ad hoc definition of the factors at the infinite place paid off and we get this eminently simple functional equation.

Remark 34. The fact that $\Lambda(L/K, \chi, s)$ admits a meromorphic continuation follows from Proposition 31. Note that this result is still new even in the abelian case becasue Theorem 27 only demonstrates the existence of an analytic continuation of $\mathcal{L}(L/K, \chi, s)$ not the complete L function. Of course, it may be easier, in the abelian case, to apply knowledge of the Γ function and its meromorphic continuation to $\mathbb{C}$ to prove the first part of the above theorem: the existence of a meromorphic continuation.

The above functional equation opens up a broad swathe of potential study, for with this functional equation comes a better understanding of the complete Artin L-function and, perhaps, this will lead to a proof of Conjecture 1 beyond just that of abelian extensions like in Theorem 27. For the nonabelian case of Conjecture 1, there are a few scattered results on low degree representations, but there is very little known in dimension above even 5 [Cog]. This is an area of active research and certainly beyond the scope of this paper.

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